Many-Valued Complete Distributivity*

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Abstract

Suppose $(\Omega, *, I)$ is a commutative, unital quantale. Categories enriched over Ω can be studied as generalized, or many-valued, ordered structures. Because many concepts, such as complete distributivity, in lattice theory can be characterized by existence of certain adjunctions, they can be reformulated in the many-valued setting in terms of categorical postulations. So, it is possible, by aid of categorical machineries, to establish theories of many-valued complete lattices, many-valued completely distributive lattices, and so on. This paper presents a systematical investigation of many-valued complete distributivity, including the topics: (1) subalgebras and quotient algebras of many-valued completely distributive lattices; (2) categories of (left adjoint) functors; and (3) the relationship between many-valued complete distributivity and properties of the quantale Ω. The results show that enriched category theory is a very useful tool in the study of many-valued versions of order-related mathematical entities.

Keywords: Commutative unital quantale, Girard quantale, quantale-enriched category, complete distributivity, subalgebra, quotient algebra.

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1 Introduction

Categories enriched over a quantale as many-valued ordered structures. Partially ordered sets are important structures in logic, mathematics, and theoretical computer sciences [1, 2, 5, 13, 30]. From the viewpoint of category theory, a partially ordered set, and generally a preordered set, is a special kind of categories, i.e., a category with hom-sets either empty or a singleton. Following Lawvere [22, 23], this fact can be put in a different way. Let $\{0,1\}$ denote the complete lattice consisting of two elements with the ordering $0 \leq 1$. Then, $\mathbf{2} = (\{0,1\}, \wedge, 1)$ is a symmetric, monoidal, closed category. Enriched categories over $\mathbf{2}$ are just the preordered sets. So, preordered sets can be investigated by aid of the categorical

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machinery. A nice example is the wide use of Galois connections, a special case of adjoint functors, in the theory of partially ordered sets. More importantly, many interesting concepts in the theory of partially ordered sets can be postulated as certain categorical properties! For example,

(1) a partially ordered set P is a lattice if and only if the diagonal $P \longrightarrow P \times P$ has both a left adjoint and a right adjoint;

(2) a partially ordered set P is a complete lattice if and only if the Yoneda embedding $\mathbf{y} : P \longrightarrow \mathcal{D}(P)$ has a left adjoint, where, $\mathcal{D}(P)$ denotes the partially ordered set of lower sets in P with the inclusion ordering and $\mathbf{y}(p) = \downarrow p = \{q \in P \mid q \leq p\}$; and

(3) a complete lattice P is (constructive) completely distributive if and only if the left adjoint of the Yoneda embedding also has a left adjoint.

The categorical postulations of lattices, complete lattices, and completely distributive lattices, can be easily reformulated for categories enriched over any arbitrary symmetric, monoidal, closed category. The aim of this paper is to study complete distributivity for categories enriched over a complete, symmetric, and monoidal closed small category, i.e., categories enriched over a commutative, unital quantale $(\Omega, *, I)$.

The study of quantale-enriched categories as generalized ordered structures originated from the theory of *quantitative domain theory*, see, e.g. [6, 12, 32, 33, 34, 38, 39]. The idea is as follows. Suppose that A is a category enriched over a commutative, unital quantale Ω. Then, for any two elements $x, y \in A$, the enrichment $A(x, y)$, an element in Ω, can be interpreted as the degree that x precedes y, or, the degree that x is smaller than or equal to y. Therefore, an Ω-category can be regarded as a *quantitative preordered set*, in which the relation between two points is expressed by an element in the quantale Ω (regarded as the set of truth-values), as opposed to the traditional *qualitative*, yes-or-no, relation in a preordered set.

Logic aspect of categories enriched over a quantale. As just mentioned in the above paragraph, in the study of categories enriched over a commutative, unital quantale Ω, the quantale can be regarded as the set of truth values. So, the study of quantale-enriched categories has a strong logical flavor. This aspect of enriched categories was emphasized by Lawvere early in 1973 in [22] as *generalized pure logic*. The idea is roughly as follows. Since Ω is a monoidal closed category, for each $\alpha \in \Omega$, the functor $\alpha * (\_) : \Omega \longrightarrow \Omega$ has a right adjoint $\alpha \to (\_) : \Omega \longrightarrow \Omega$. That means, for all $\alpha, \beta, \gamma \in \Omega$,

$$\alpha * \beta \leq \gamma \iff \beta \leq \alpha \to \gamma.$$

Therefore, if we interpret α, β, and γ as truth values, then the operation $*$ plays a similar role as the logic connective *conjunction* and $\to$ as the connective *implication*. The least element $0 \in \Omega$ can be regarded as the logical value *absurdity* and the unit element I as the value *true*. Moreover, for every set X, a function $\lambda : X \longrightarrow \Omega$ can be regarded as a predicate on X, the element $\lambda(x)$ is the degree that x has certain attribute. And it is natural to interpret $\bigwedge_{x \in X} \lambda(x)$ and $\bigvee_{x \in X} \lambda(x)$ as the truth degree for the logical formulas $\forall x \lambda$ and $\exists x \lambda$ respectively. These observations

relate the study of quantale-enriched categories to many-valued logic.

By a "many-valued logic" we mean a logic of which the truth-value set is just a commutative, unital quantale. Such a logic is called a *Monoidal Logic* in [17, 10]. When the quantale Ω is a BL-algebra, this kind of logic has been extensively investigated under the name *Basic Logic* in the literature, see, e.g. Hájek [15, 16]; and when Ω is a commutative Girard quantale, such a logic is a commutative version of *Linear Logic* initiated by Girard, see, e.g. [14, 30, 41]. And, if Ω is at the same time a BL-algebra and a commutative Girard quantale, then Ω must be an MV-algebra [17], in this case we come back to the *Many-Valued Logic* initiated by Łukasiewicz [9]. However, by abuse of language, we shall call a "monoidal logic" simply a "many-valued logic" in this paper because that the truth value set is not a boolean algebra in general and hence contains more than two elements. And so the title of this paper.

Relationship to constructive complete distributivity. It is well-known that there already exists a notion of lattice in any topos [24, 40]. Since many concepts in lattice theory can be postulated as categorical properties, they can be easily reformulated in any topos. So, we can establish theories of complete lattices, completely distributive lattices, and so on, within any topos. For complete distributivity, this has already been done under the name *constructive complete distributivity* in a series of papers [11, 28, 29, 40]. These theories are developed within the framework of the internal logic in a topos.

But, many-valued complete distributivity could be regarded as, to some extent, a mathematical theory developed within the framework of an *observed* logic. This can be roughly explained as follows. Suppose that we are working in **Set**, even with the Axiom of Choice allowed if you prefer. Contemplation over principles of our reasoning about mathematical entities, or generally the states of affairs around us, leads us to observe that not only the Boolean algebra **2**, but also any MV-algebra, any BL-algebra, any commutative unital quantale, posses sufficient structures to act as a *ruler* or a *criterion* in our reasoning about "states of affairs in reality" [1]. Then, if we, tempted by the fun to know, take this kind of structures as our criteria[2], or truth values, we are led to theories of many-valued logics. Many-valued complete distributivity discussed in this paper is a mathematical theory developed within the framework of an observed logic with a commutative, unital quantale as the set of truth values. This method could be generalized to establish many-valued versions of other mathematical entities.

Since both the notion of many-valued complete distributivity and that of constructive complete distributivity are postulated as a certain categorical property, categorical methods play an essential role in the study of both of these notions. And more interestingly, many-valued complete distributivity and constructive complete distributivity have similar properties. The reader can compare the following Theorem 1.1 and Theorem 1.2.

[1]This phrase is quoted from Dirk van Dalen: Logic and Structure, 4th edition, Springer, 2004.
[2]Sometimes, this is necessary and helpful as exemplified in [9, 15].

Theorem 1.1 *Suppose that Ω is an integral commutative quantale. Then the following conditions are equivalent:*

(1) *Ω is a Girard quantale, i.e., it satisfies the law of double negation.*

(2) *The dual of every completely distributive Ω-lattice is also a completely distributive Ω-lattice.*

(3) *Ω^{op} is a many-valued Heyting algebra.*

Theorem 1.2 [28, 40] *In any topos $\mathcal{E}$ the following conditions are equivalent:*

(1) *$\mathcal{E}$ is Boolean.*

(2) *The dual of every constructive completely distributive lattice is constructive completely distributive.*

(3) *The dual lattice of the truth value set is a Heyting algebra.*

However, the exact relationship between many-valued complete distributivity and constructive complete distributivity still awaits further investigation.

Related works. Categories enriched over a commutative, unital quantale Ω have received wide attention in the literature since the publication of the pioneering paper of Lawvere [22]. The order aspect of Ω-categories leads the theory of quantitative domains, see, e.g. [6, 12, 32, 34, 38, 39]. These works have a strong background in theoretical computer sciences, so most of them are concerned with certain kind of directed completeness of Ω-categories. A systematical investigation of directed completeness for Ω-categories is presented in [21]. And it should be pointed out that the order aspect of quantale-enriched categories has also been studied under the name *fuzzy order*, see, e.g. [3, 4].

Most of the results in Section 3 on characterization of completeness of Ω-categories are special cases of the general results for enriched categories in [7, 19]. Because of the simplicity of Ω, these results appear in an extremely simple form. The equivalence between cocomplete Ω-categories and Ω-modules [2, 18, 31] was first observed by Stubbe [35].

The notion of completely distributive Ω-lattices was introduced in [37] and [42]. Stubbe has obtained many interesting results about many-valued complete distributivity in [37]. It is necessary to make clear the relationship between [37] and this paper. At first, [37] focuses on characterizations of many-valued complete distributivity in the category of cocomplete Ω-categories and cocontinuous Ω-functors. This paper is, with emphasis on the order aspect of completely distributive Ω-lattices, dealing with the category of many-valued completely distributive Ω-lattices and complete Ω-lattice morphisms. Secondly, the approach in [37] is comparatively more *sophisticated*, it depends heavily on the computation techniques on weighted limits and weighted colimits developed in [36] for quantaloids. But, the approach in this paper is quite *elementary*. Thirdly, the most important, except propositions 4.3 and 4.5 (with different proofs), there is little overlap between the results about many-valued completely distributive lattices in the article [37] and this one.

And, it should be pointed out that in [37] Ω is not assumed to be commutative.

One problem with the absence of commutativity is that the dual of an Ω-category is not an Ω-category in general (Example 2.3). Though many results and proofs in this paper can be improved to cope with the absence of commutativity of Ω as in [37], we have assumed the commutativity of Ω in order to be succinct.

Summary of the contents. This paper is devoted to a systematical investigation of many valued complete distributivity. The contents are arranged as follows.

In section 2, basic notions of Ω-categories and complete Ω-lattices are recalled.

In section 3, some equivalent descriptions of complete Ω-lattices are given. Of particular interest is the observation by Stubbe [35] that complete Ω-lattices are essentially Ω-modules. These characterizations of complete Ω-lattices shall be often employed in the following sections.

Section 4 introduces the notion of many-valued completely distributive lattices, i.e., completely distributive Ω-lattices, and discusses some basic properties of these objects.

Section 5 focuses on the subalgebras and quotient algebras of a completely distributive Ω-lattice. The main result says that the subalgebras of a completely distributive Ω-lattice A correspond bijectively to the cocontinuous closure operators on A and the quotient algebras of a completely distributive Ω-lattice A correspond bijectively to the cocontinuous kernel operators on A.

Section 6 is an application of the results in Section 5. It is proved that the category of left adjoints between completely distributive Ω-lattices is completely distributive by showing that it is a quotient algebra of some completely distributive Ω-lattice.

The last section, Section 7, deals with the question that whether the dual of a completely distributive Ω-lattice is also completely distributive. The main result in this section is Theorem 1.1 stated in the above. This result relates many-valued complete distributivity closely to properties of the truth-value set Ω.

2 Complete Ω-lattices

We refer to [7, 25] for general category theory, to [7, 19, 22] for enriched category theory, and to [5, 13] for lattice theory.

Let Ω be a complete lattice. The greatest element of Ω is denoted 1 and the least element of Ω is denoted 0. For $U \subset \Omega$, write $\bigvee U$ for the least upper bound of U and $\bigwedge U$ for the greatest lower bound of U. Particularly, $\bigvee \emptyset = 0$ and $\bigwedge \emptyset = 1$.

A commutative, unital quantale is a triple $(\Omega, I, *)$, abbreviated as Ω, where, Ω is a complete lattice, I is a fixed element in Ω, and $* : \Omega \times \Omega \longrightarrow \Omega$, called the tensor, is a commutative, associative binary operation such that (1) $*$ is monotone on each variable; (2) I is a unit element for $*$, i.e. $\alpha * I = \alpha$ for every $\alpha \in \Omega$; and (3) for each $\alpha \in \Omega$, the monotone function $\alpha * (\_) : \Omega \longrightarrow \Omega$ has a right adjoint $\alpha \rightarrow (\_) : \Omega \longrightarrow \Omega$. The resulting binary operation $\rightarrow: \Omega \times \Omega \longrightarrow \Omega$ is called the residuation operator, or implication, corresponding to the tensor $*$.

Throughout this paper, $(\Omega, *, I)$ will always denote a commutative, unital quantale. And when there will be no confusion with respect to the tensor $*$ and the unit I, we often write simply Ω instead of $(\Omega, *, I)$. Some basic properties of the tensor operator and residuation operator are collected in the following, most of them can be found in many places, for instance, [3, 15, 17, 30].

Proposition 2.1 (1) $0 * \alpha = 0$ for all $\alpha \in \Omega$.

(2) $\alpha \to \beta = \bigvee\{\gamma : \alpha * \gamma \leq \beta\}$.

(3) $I \to \alpha = \alpha$; $0 \to \alpha = 1$ for all $\alpha \in \Omega$.

(4) $(\alpha \to \beta) * (\beta \to \gamma) \leq (\alpha \to \gamma)$.

(5) $\alpha \to (\beta \to \gamma) = (\alpha * \beta) \to \gamma = \beta \to (\alpha \to \gamma)$.

(6) $((\alpha \to \beta) \to \beta) \to \beta = \alpha \to \beta$.

(7) $\alpha * \bigvee_{j \in J} \beta_j = \bigvee_{j \in J} \alpha * \beta_j$.

(8) $(\bigvee_{j \in J} \alpha_j) \to \beta = \bigwedge_{j \in J}(\alpha_j \to \beta)$; $\alpha \to (\bigwedge_{j \in J} \beta_j) = \bigwedge_{j \in J}(\alpha \to \beta_j)$.

(9) $\bigwedge_{\gamma \in \Omega} \Big((\gamma \to \alpha) \to (\gamma \to \beta)\Big) = \alpha \to \beta$.

(10) $\bigwedge_{\gamma \in \Omega} \Big((\alpha \to \gamma) \to (\beta \to \gamma)\Big) = \beta \to \alpha$.

If the unit I coincides with the greatest element 1 in Ω, Ω is called an *integral commutative quantale*, or, a *complete residuated lattice*. An integral commutative quantale Ω is called a *complete BL-algebra* [15] if it satisfies

(11) $\alpha * (\alpha \to \beta) = \alpha \wedge \beta$ (divisibility); and

(12) $(\alpha \to \beta) \vee (\beta \to \alpha) = 1$ (prelinearity).

BL-algebras are the algebras for the *Basic Logic* developed in [15].

A commutative unital quantale Ω is called a *commutative Girard quantale* if it satisfies the *law of double negation*:

(13) $\alpha = (\alpha \to 0) \to 0$.

This definition of commutative Girard quantale is taken from Höhle [17] and it is stronger than the definition in [30]. A commutative Girard quantale Ω is necessarily integral since

$$I = (I \to 0) \to 0 = 0 \to 0 = 1.$$

Girard quantales are closely related to the *Linear Logic* developed by Girard [14].

A *complete MV-algebra* is a commutative unital quantale which is simultaneously a BL-algebra and a Girard quantale [17]. For a nice exposition of MV-algebras and their role in many-valued logic, we refer to the monograph [9].

A category enriched over a commutative quantale Ω, or an Ω-category, is a pair $(A, \hom)$ with A a set and hom a function assigning to every ordered pair of $(a, b) \in A \times A$ an element $\hom(a, b) \in \Omega$, such that (1) $I \leq \hom(a, a)$ for every $a \in A$ (reflexivity); and (2) $\hom(a, b) * \hom(b, c) \leq \hom(a, c)$ for all $a, b, c \in A$ (transitivity).

In an Ω-category $(A, \hom)$, A is called the underlying set of $(A, \hom)$ and the

function hom is called the *hom functor*. We often write simply A for an Ω-category and $A(x, y)$ for $\hom(x, y)$ if the hom functor is clear from the context. And in this case we write $|A|$ for the underlying set of A.

An Ω-functor between Ω-categories A and B is a function $f : A \longrightarrow B$ such that $A(a, b) \leq B(f(a), f(b))$ for all $a, b \in A$. An Ω-functor f is called an Ω-*isometry* if $A(a, b) = B(f(a), f(b))$ for all $a, b \in A$. If an Ω-isometry f is also bijective on the underlying sets, it will be called an Ω-*isomorphism*, or an isomorphism for short. Ω-functors are composed by composing the underlying functions on sets.

An Ω-category A can also be regarded as an Ω-valued preordered set with the value $A(x, y) \in \Omega$ being interpreted as the degree to which x is smaller than or equal to y. An Ω-functor is also called an Ω-*monotone function* since the condition $A(a, b) \leq B(f(a), f(b))$ asserts that if $a \leq b$ in A, then $f(a) \leq f(b)$ in B. In this paper we switch freely between the terms of Ω-categories and Ω-preordered sets, and between Ω-functors and Ω-monotone functions. When we want to emphasize the categorical aspect of A, we say A is an Ω-category; and when we want to emphasize the order aspect of A, we say A is an Ω-preordered set. And so for the terms Ω-functors and Ω-monotone functions.

Suppose A is an Ω-category. We define a binary relation $\leq$ on the underlying set of A in the following way: $a \leq b$ if $A(a, b) \geq I$. It is easily seen that $\leq$ is a preorder, i.e. a reflexive and transitive relation, on $|A|$. For each Ω-category, we write A_0 for the preordered set $(|A|, \leq)$. In this way, we obtain a forgetful functor $(\_)_0 : \Omega\text{-}\mathbf{Cat} \longrightarrow \mathbf{PrOrd}$ from the category Ω-**Cat** of Ω-categories to the category **PrOrd** of preordered sets.

Two elements x and y in an Ω-category A are said to be *isomorphic* if $A(x, y) \geq I$ and $A(y, x) \geq I$. An Ω-category A is called *antisymmetric* if different elements in A are always non-isomorphic, or equivalently, A_0 is a partially ordered set. An anti-symmetric Ω-category is also called an *partially* Ω-*ordered set*.

In the following examples we list some methods to construct Ω-categories. These methods are somewhat standard in category theory and it is hard to find where they appeared for the first time, so, we don't include any reference here.

Examples 2.2 (1) (The canonical Ω-category structure on Ω) Let $\Omega(\alpha, \beta) = \alpha \rightarrow \beta$. Then, by Proposition 2.1, it is easy to check that Ω is a partially Ω-ordered set.

(2) (Discrete Ω-categories) Given a set X and $x, y \in X$, let $X(x, y) = I$ if $x = y$ and $X(x, y) = 0$ if $x \neq y$. Then X becomes an Ω-category. Such Ω-categories are called discrete since that for any Ω-category B, every function from X to B is an Ω-functor. We write $\mathbf{1}$ for the discrete Ω-category consisting of exactly one element.

(3) (Terminal object) Let $X = \{x\}$ be a singleton and $X(x, x) = 1$, the top element in Ω. Then X is an Ω-category and it is the terminal object in the category of Ω-categories, which shall be denoted $\top$ in the sequel.

(4) (Dual Ω-category) Suppose A is an Ω-category. Let $A^{\mathrm{op}}(a, b) = A(b, a)$ for all $a, b \in A$. Then A^{op} is also an Ω-category, called the dual of A.

(5) (Subcategory) Let A be an Ω-category and B is a subset of A. For all $x, y \in B$,

let $B(x,y)=A(x,y)$. Then B becomes an Ω-category, called a *subcategory* of A.

(6) (Product category) Suppose $\{A_i : i \in J\}$ is a family of Ω-categories, the product of $\{A_i : i \in J\}$ in the category Ω-**PrOrd** is given by

$$\prod_{i\in J} A_i(a,b) = \bigwedge_{i\in J} A_{i\in J}(a_i,b_i),\ a=(a_i)_{i\in I},\ b=(b_i)_{i\in J}.$$

(7) (Functor category) Given Ω-categories A and B, denote the set of all the Ω-functors from A to B by $[A,B]$. Let $[A,B](f,g) = \bigwedge_{x\in A} B(f(x),g(x))$ for all $f,g \in [A,B]$. Then $[A,B]$ becomes an Ω-category, called the *functor category* from A to B.

If X is a discrete Ω-category, then $[X,B]$ consists of all the function from X to B. Particularly, given an Ω-category A, let $|A|$ denote the Ω-category obtained by equipping the underlying set of A with the discrete Ω-categorical structure. Sometimes, we write $[\Omega^A]$ for$[|A|,\Omega]$ in the sequel. Clearly, both $[A^{\rm op},\Omega]$ and $[A,\Omega]$ are subcategories of $[\Omega^A]$.

Example 2.3 Suppose that $(\Omega,*,I)$ is a unital quantale, i.e., $(\Omega,*,I)$ satisfies the conditions for a commutative unital quantale except, possibly, the commutativity of the binary operation $*$. For all $a,b\in\Omega$, let $\Omega(a,b) = a \rightarrow b = \bigvee\{r\in\Omega : a*r \leq b\}$. Then $(\Omega,\rightarrow)$ becomes an Ω-category. However, the dual $\Omega^{\rm op}$ of $(\Omega,\rightarrow)$, given by $\Omega^{\rm op}(a,b) = b\rightarrow a$, is an Ω-category if and only if Ω is commutative. In fact, if $\Omega^{\rm op}$ is an Ω-category, then for all $a,b\in\Omega$,

$$b*a \leq [a\rightarrow(a*b)]*a = \Omega^{\rm op}(a*b,a)*\Omega^{\rm op}(a,I) \leq \Omega^{\rm op}(a*b,I) = a*b.$$

Exchanging the role of a and b, we have that $a*b \leq b*a$. Thus, Ω is commutative.

Definition 2.4 Given an Ω-category A, the Yoneda embedding is the function $\mathbf{y} : A \longrightarrow [A^{\rm op},\Omega]$ given by $\mathbf{y}(a)(x) = A(x,a)$ and the co-Yoneda embedding is the function $\mathbf{y}' : A \longrightarrow [A,\Omega]^{\rm op}$ given by $\mathbf{y}'(a)(x) = A(a,x)$.

The following lemma shows that both the Yoneda embedding and the co-Yoneda embedding are Ω-isometries.

Lemma 2.5 (Yoneda) (1) *For all $a\in A$ and $\phi\in[A^{\rm op},\Omega]$, $[A^{\rm op},\Omega](\mathbf{y}(a),\phi) = \phi(a)$.*

(2) *For all $a\in A$ and $\psi\in[A,\Omega]$, $[A,\Omega](\mathbf{y}'(a),\psi) = \psi(a)$.* $\Box$

If A is a (classical) preordered set. Then a $\mathbf{2}$-functor $\psi : A \longrightarrow \mathbf{2}$ is, regarded as a characteristic function, precisely an upper set of A and a $\mathbf{2}$-functor $\phi\in[A^{\rm op},\mathbf{2}]$ precisely a lower set of A. Thus, an Ω-functor $\psi\in[A,\Omega]$ shall also be called an upper Ω-subset and $\phi\in[A^{\rm op},\Omega]$ a lower Ω-subset in A. Particularly, $\mathbf{y}(a) = A(-,a)$ and $\mathbf{y}'(a) = A(a,-)$ are called respectively the principal lower Ω-subset and the principal upper Ω-subset of A generated by a.

Proposition 2.6 ([21]) *Let A be an Ω-category. Then for all $\alpha\in\Omega$, $\mathcal{F}\subset[A,\Omega]$, and $\psi\in[A,\Omega]$, the followings hold:*

(1) $\bigvee \mathcal{F} \in [A, \Omega]$. *Particularly, as the join of an empty family,* $0_A \in [A, \Omega]$.

(2) $\bigwedge \mathcal{F} \in [A, \Omega]$. *Particularly, as the meet of an empty family,* $1_A \in [A, \Omega]$.

(3) $\alpha * \psi \in [A, \Omega]$.

(4) $\alpha \to \psi \in [A, \Omega]$.

(5) $\psi \to \alpha \in [A^{\mathrm{op}}, \Omega]$ *and* $\psi = \bigwedge_{\alpha \in \Omega} \Big((\psi \to \alpha) \to \alpha \Big)$. $\square$

For each $\mu \in \Omega^A$ and $x \in A$, let

$$(\uparrow \mu)(x) = \bigvee_{y \in A} \mu(y) * A(y, x) = \bigvee_{y \in A} \mu(y) * \mathbf{y}'(y)(x),$$

and

$$(\downarrow \mu)(x) = \bigvee_{y \in A} \mu(y) * A(x, y) = \bigvee_{y \in A} \mu(y) * \mathbf{y}(y)(x).$$

It is easy to verify that $\uparrow \mu \in [A, \Omega]$ and $\uparrow \mu$ is the smallest upper Ω-subset in A which is bigger than or equal to μ under the pointwise order. Dually, $\downarrow \mu$ is the smallest lower Ω-subset in A which is bigger than or equal to μ under the pointwise order. Particularly, $\phi \in [A, \Omega] \iff \uparrow \phi = \phi$ and $\phi \in [A^{\mathrm{op}}, \Omega] \iff \downarrow \phi = \phi$. And we left it to the reader to check that the two operations $\uparrow$ and $\downarrow$ are Ω-functors, i.e. $\uparrow \colon [\Omega^A] \longrightarrow [A, \Omega]$ and $\downarrow \colon [\Omega^A] \longrightarrow [A^{\mathrm{op}}, \Omega]$ are Ω-functors.

The following definition is a special case of the general concept of enriched adjunctions in the theory of enriched categories [7, 19].

Definition 2.7 [38, 39] A pair of Ω-functors $f : A \longrightarrow B$ and $g : B \longrightarrow A$ is said to be an Ω-adjunction (or simply an adjunction) between Ω-categories A and B, if $B(f(a), b) = A(a, g(b))$ for all $a \in A$ and $b \in B$. In this case, we say f is a left adjoint of g and g is a right adjoint of f.

It is easily seen that if (f, g) is an Ω-adjunction between Ω-categories A and B, then the pair of monotone functions $f : A_0 \longrightarrow B_0$ and $g : B_0 \longrightarrow A_0$ is a Galois connection, i.e. a $\mathbf{2}$-adjunction, between the preordered set A_0 and B_0.

Proposition 2.8 ([21]) *Suppose A and B are Ω-categories and $f : A \longrightarrow B$, $g : B \longrightarrow A$ are functions. Then the following conditions are equivalent:*

(1) *(f, g) is an Ω-adjunction.*

(2) *f is an Ω-functor and $B(f(a), b) = A(a, g(b))$ for all $a \in A$ and $b \in B$.*

(3) *g is an Ω-functor and $B(f(a), b) = A(a, g(b))$ for all $a \in A$ and $b \in B$.*

These conditions imply

(4) *If (f, g) is an Ω-adjunction between Ω-categories A and B, then $f \circ g \circ f = f$ and $g \circ f \circ g = g$.* $\square$

Proposition 2.9 *If (f, g) is an Ω-adjunction between Ω-categories A and B, then*

(1) *The following conditions are equivalent:*

(a) f is injective,

(b) $g \circ f = \mathrm{id}_A$, and

(c) g is surjective.

In this case, f is an isometry.

(2) The following conditions are equivalent:

(a′) f is surjective,

(b′) $f \circ g = \mathrm{id}_B$, and

(c′) g is injective.

In this case, g is an isometry. $\square$

Let $f : A \longrightarrow B$ be an Ω-monotone function. For each $\psi \in [B, \Omega]$, let $f^{\leftarrow}(\psi) = \psi \circ f$. Then we obtain an Ω-functor $f^{\leftarrow} : [B, \Omega] \longrightarrow [A, \Omega]$.

Proposition 2.10 [22] *The Ω-functor $f^{\leftarrow} : [B, \Omega] \longrightarrow [A, \Omega]$ has both left and right adjoints. The left adjoint is given by $f_\ell^{\rightarrow} : [A, \Omega] \longrightarrow [B, \Omega]$, where for all $\psi \in [A, \Omega]$,*

$$f_\ell^{\rightarrow}(\psi)(y) = \bigvee_{x \in A} \psi(x) * B(f(x), y).$$

The right adjoint is given by $f_r^{\rightarrow} : [A, \Omega] \longrightarrow [B, \Omega]$, where for all $\psi \in [A, \Omega]$,

$$f_r^{\rightarrow}(\psi)(y) = \bigwedge_{x \in A} B(y, f(x)) \rightarrow \psi(x). \quad \square$$

Categorically, $f_\ell^{\rightarrow}(\psi)$ and $f_r^{\rightarrow}(\psi)$ are the Kan-extensions of ψ along f [7, 22].

Similarly, for each $\phi_B \in [B^{\mathrm{op}}, \Omega]$ and $\phi_A \in [A^{\mathrm{op}}, \Omega]$, let $f^{\leftarrow}(\phi_B) = \phi_B \circ f \in [A^{\mathrm{op}}, \Omega]$ and

$$f_\ell^{\rightarrow}(\phi_A)(y) = \bigvee_{x \in A} \phi_A(x) * B(y, f(x))$$

for all $y \in B$. Then both $f^{\leftarrow} : [B^{\mathrm{op}}, \Omega] \longrightarrow [A^{\mathrm{op}}, \Omega]$ and $f_\ell^{\rightarrow} : [A^{\mathrm{op}}, \Omega] \longrightarrow [B^{\mathrm{op}}, \Omega]$ are Ω-functors.

Suppose that $f : A \longrightarrow B$ is an Ω-functor and $\phi : A \longrightarrow \Omega$ is a function (not necessarily an Ω-functor). For each $b \in B$, let

$$f(\phi)(b) = \bigvee_{x \in f^{-1}(b)} \phi(x).$$

Then $f(\phi)$ is called the image of ϕ under f. We left it to the reader to check that for any $\psi \in [A, \Omega]$, $f_\ell^{\rightarrow}(\psi) = \uparrow f(\psi)$ and for any $\phi \in [A^{\mathrm{op}}, \Omega]$, $f_\ell^{\rightarrow}(\phi) = \downarrow f(\phi)$.

Example 2.11 Let A be an Ω-category. If we write $i : [A, \Omega] \longrightarrow [\Omega^A]$ for the inclusion functor. Then $(\uparrow, i)$ is an Ω-adjunction. Similarly, if we write $i : [A^{\mathrm{op}}, \Omega] \longrightarrow [\Omega^A]$ for the inclusion functor. Then $(\downarrow, i)$ is also an Ω-adjunction.

This fact can be put as a special case of the above proposition. Let $|A|$ denote the underlying set of A with the discrete Ω-category structure. Then, the identity

function $\mathrm{id} : |A| \longrightarrow A$ is an Ω-functor. Clearly, $\mathrm{id}^{\leftarrow} : [A^{\mathrm{op}}, \Omega] \longrightarrow [\Omega^A]$ is exactly the inclusion $i : [A^{\mathrm{op}}, \Omega] \longrightarrow [\Omega^A]$ and $\mathrm{id}_\ell^{\rightarrow}(\phi) = {\downarrow}\phi$ for all $\phi \in \Omega^A$. Hence, $(\downarrow, i)$ is an Ω-adjunction.

Definition 2.12 [21, 38, 39] An Ω-category A is called complete if the co-Yoneda embedding $\mathbf{y}' : A \longrightarrow [A, \Omega]^{\mathrm{op}}$ has a right adjoint $\inf : [A, \Omega]^{\mathrm{op}} \longrightarrow A$. And A is called cocomplete if the Yoneda embedding $\mathbf{y} : A \longrightarrow [A^{\mathrm{op}}, \Omega]$ has a left adjoint $\sup : [A^{\mathrm{op}}, \Omega] \longrightarrow A$.

Cocomplete Ω-categories are a special case of the *total* (enriched) categories in [20]. However, because of the simplicity of Ω and the fact that we don't have size problems here, properties of these Ω-categories will become much simpler and more elegant.

Proposition 2.13 [42] *For an Ω-category A, the following are equivalent:*

(1) *A is complete.*

(2) *The composition functor $i \circ \mathbf{y}' : A \longrightarrow [A, \Omega]^{\mathrm{op}} \longrightarrow [\Omega^A]^{\mathrm{op}}$ has a right adjoint.*

(3) *A is cocomplete.*

(4) *The composition functor $i \circ \mathbf{y} : A \longrightarrow [A^{\mathrm{op}}, \Omega] \longrightarrow [\Omega^A]$ has a left adjoint.*

□

That $i \circ \mathbf{y} : A \longrightarrow [A^{\mathrm{op}}, \Omega] \longrightarrow [\Omega^A]$ has a left adjoint amounts to say that for each $\phi \in [\Omega^A]$, there is an element $a \in A$ such that for all $x \in A$,

$$A(a, x) = [\Omega^A](\phi, \mathbf{y}(x)) = \bigwedge_{z \in A} (\phi(z) \to A(z, x)).$$

The condition that for all $x \in A$, $A(a, x) = \bigwedge_{z \in A} (\phi(z) \to A(z, x))$, can be interpreted as the statement that for all $x \in A$, a is smaller than or equal to x if and only if ϕ is contained in the principal lower Ω-subset generated by x. In other words, a is the supremum of ϕ in A [38, 39].

Generally, we say that a function $\phi : A \longrightarrow \Omega$ has a supremum if there is some element (unique up to isomorphisms) $a \in A$ such that for all $x \in A$, $A(a, x) = \bigwedge_{z \in A} (\phi(z) \to A(z, x))$. Then, by Proposition 2.8, an Ω-category A is cocomplete if and only if every function $\phi : A \longrightarrow \Omega$ has a supremum in A.

If A is cocomplete, the left adjoint of $i \circ \mathbf{y} : A \longrightarrow [\Omega^A]$ is given by $\sup \circ \downarrow : [\Omega^A] \longrightarrow [A^{\mathrm{op}}, \Omega] \longrightarrow A$. Thus, for each $\phi \in [\Omega^A]$, the supremum of ϕ is $\sup({\downarrow}\phi)$. Thus, we shall write simply $\sup \phi$ for the supremum of ϕ in the sequel.

Similarly, we can define the *infimum* $\inf \phi$ $(= \inf({\uparrow}\phi))$ of a function $\phi : A \longrightarrow \Omega$ in A and obtain similar results.

Example 2.14 Both the singleton discrete Ω-category $\mathbf{1}$ and the terminal Ω-category $\top$ are complete.

Proposition 2.15 ([7]) *For each complete Ω-category A, the underlying preordered set A_0 of A is complete.*

A function $\mu : A \longrightarrow \Omega$ is said to be a finite Ω-subset if the set $\{a \in A : \mu(a) \neq 0\}$ is finite.

Definition 2.16 An Ω-functor $f : A \longrightarrow B$ is said to preserve (finite) sups if $f(\sup{}_A\mu) = \sup{}_B f(\mu)$ whenever $\sup{}_A\mu$ exists for any (finite) $\mu \in [\Omega^A]$; and f is said to preserve (finite) infs if $f(\inf{}_A\mu) = \inf{}_B f(\mu)$ whenever $\inf{}_A\mu$ exists for any (finite) $\mu \in [\Omega^A]$. f is also said to be continuous if it preserves infs and cocontinuous if it preserves sups.

The following theorem is a generalization of the properties of Galois connections between complete lattices. It relates the existence of left (right) adjoints to the preservation of infima (suprema resp.).

Theorem 2.17 ([21, 35]) *Let $f : A \longrightarrow B$ and $g : B \longrightarrow A$ are Ω-functors. If (f, g) is an Ω-adjunction, then $f : A \longrightarrow B$ preserves sups and $g : B \longrightarrow A$ preserves infs. Conversely, if A is cocomplete then f has a right adjoint whenever f preserves sups; and if B is complete then g has a left adjoint whenever g preserves infs.* $\square$

3 Equivalents of complete Ω-lattices

Definition 3.1 ([7, 19]) An Ω-category A is said to be tensored if for all $\alpha \in \Omega, x \in A$, there is an element $\alpha \otimes x \in A$, called the tensor of α and x, such that $A(\alpha \otimes x, y) = \alpha \to A(x, y)$ for any $y \in A$. A is said to be cotensored if, for all $\alpha \in \Omega$ and $y \in A$, there is some $\alpha \rightarrowtail x \in A$, called the cotensor of α and x, such that $A(z, \alpha \rightarrowtail x) = \alpha \to A(z, x)$ for any $z \in A$.

By definition, every complete (hence cocomplete) Ω-category is both tensored and cotensored. And it is easily seen that the tensor of α and x in A is just the cotensor of α and x in A^{op}. Since left adjoints preserve suprema, they preserve tensors in the sense that $f(\alpha \otimes x) = \alpha \otimes f(x)$, see Proposition 3.6 for a proof. Dually, right adjoints preserve cotensors.

Examples 3.2 (1) $(\Omega, \to)$ is tensored and cotensored. Indeed, $\alpha \otimes x = \alpha * x$ and $\alpha \rightarrowtail x = \alpha \to x$.

(2) For every Ω-category A, the functor category $[A^{\mathrm{op}}, \Omega]$ is tensored and cotensored. For all $\lambda \in [A^{\mathrm{op}}, \Omega]$ and $\alpha \in \Omega$, the tensor $\alpha \otimes \lambda$ of α and λ in $[A^{\mathrm{op}}, \Omega]$ is $\alpha * \lambda$ and the cotensor $\alpha \rightarrowtail \lambda$ of α and λ in $[A^{\mathrm{op}}, \Omega]$ is $\alpha \to \lambda$. Similarly, the tensor and cotensor of $\alpha \in \Omega$ and $\mu \in [A, \Omega]$ in $[A, \Omega]$ are given by $\alpha * \mu$ and $\alpha \to \mu$ respectively.

Definition 3.3 A complete Ω-lattice is an antisymmetric, complete (hence cocomplete) Ω-category.

Convention: Suppose A is a complete Ω-lattice. Then A_0 is a complete lattice. Given a subset $(x_t)_{t\in T}$ of A, the least upper bound of $(x_t)_{t\in T}$ in the complete lattice A_0 is called the *join* of $(x_t)_{t\in T}$, $\bigvee_{t\in T} x_t$ in symbols; and the greatest lower bound

of $(x_t)_{t\in T}$ in A_0 is called the *meet* of $(x_t)_{t\in T}$, $\bigwedge_{t\in T} x_t$ in symbols. We reserve the notations sup and inf for supremum and infimum in A. That is, for every function $\phi : A \longrightarrow \Omega$, $\sup\phi$ stands for the *supremum* of ϕ in A, and $\inf\phi$ for the *infimum* of ϕ in A. So, for example, given a function $\phi : A \longrightarrow \Omega$, $\bigvee_{x\in A}\phi(x)$ denotes the least upper bound of $\{\phi(x) \mid x \in A\}$ in the complete lattice Ω; meanwhile $\sup\phi$ is an element in A, the supremum of ϕ in A.

Proposition 3.4 *Suppose A is a tensored and cotensored Ω-category such that A_0 is a complete lattice. Then, the tensor $\otimes : \Omega \times A_0 \longrightarrow A_0$ and cotensor $\rightarrowtail : \Omega\otimes A_0 \longrightarrow A_0$ satisfy the following conditions:*

(1) $A(\alpha\otimes x, y) = \alpha \to A(x,y) = A(x, \alpha \rightarrowtail y)$. *Hence* $\alpha\otimes x \le y \iff \alpha \le A(x,y) \iff x \le \alpha\rightarrowtail y$.

(2) (i) $I\otimes x = x$; (i′) $I\rightarrowtail x = x$.

(3) (ii) $(\alpha * \beta)\otimes x = \alpha\otimes(\beta\otimes x)$; (ii′) $(\alpha*\beta)\rightarrowtail x = \alpha\rightarrowtail(\beta\rightarrowtail x)$.

(4) *For any $x \in A$, the function $(\_)\otimes x : \Omega \longrightarrow A_0$ is a left adjoint of the function $A(x,-) : A_0 \longrightarrow \Omega$. Hence,*

$$\text{(iii)}\ \Big(\bigvee_{t\in T}\alpha_t\Big)\otimes x = \bigvee_{t\in T}(\alpha_t\otimes x);\quad \text{(iii}'\text{)}\ A\Big(x, \bigwedge_{t\in T} x_t\Big) = \bigwedge_{t\in T} A(x, x_t).$$

(5) *For any $x \in A$, the function $(\_)\rightarrowtail x : \Omega \longrightarrow A_0^{\mathrm{op}}$ is a left adjoint of the function $A(-,x) : A_0^{\mathrm{op}} \longrightarrow \Omega$. Hence,*

$$\text{(iv)}\ A\Big(\bigvee_{t\in T} x_t, x\Big) = \bigwedge_{t\in T} A(x_t, x);\quad \text{(iv}'\text{)}\ \Big(\bigvee_{t\in T}\alpha_t\Big)\rightarrowtail x = \bigwedge_{t\in T}(\alpha_t\rightarrowtail x).$$

(6) *For any $\alpha \in \Omega$, the function $\alpha\otimes(\_) : A_0 \longrightarrow A_0$ is a left adjoint of the function $\alpha\rightarrowtail(\_) : A_0 \longrightarrow A_0$. Hence,*

$$\text{(v)}\ \alpha\otimes\Big(\bigvee_{t\in T} x_t\Big) = \bigvee_{t\in T}\alpha\otimes x_t;\quad \text{(v}'\text{)}\ \alpha\rightarrowtail\Big(\bigwedge_{t\in T} x_t\Big) = \bigwedge_{t\in T}(\alpha\rightarrowtail x_t).$$

Proof. (1) and (2) follows from definition immediately.

(3) For any $x, y \in A$,

$$\begin{aligned} A(\alpha\otimes(\beta\otimes x), y) &= \alpha \to A(\beta\otimes x, y) = \alpha\to(\beta\to A(x,y)) \\ &= (\alpha*\beta)\to A(x,y) = A((\alpha*\beta)\otimes x, y), \end{aligned}$$

and

$$\begin{aligned} A(x, \alpha\rightarrowtail(\beta\rightarrowtail y) &= \alpha\to A(x, \beta\rightarrowtail x) = \alpha\to(\beta\to A(x,y)) \\ &= (\alpha*\beta)\to A(x,y) = A(x, (\alpha*\beta)\rightarrowtail y). \end{aligned}$$

(4), (5) and (6) follow from (1) straightforwardly. $\square$

Proposition 3.5 *Suppose A and B are complete Ω-lattices. Then $f : A \longrightarrow B$ is an Ω-functor if and only if (1) $f : A_0 \longrightarrow B_0$ preserves order and (2) $\alpha \otimes f(x) \leq f(\alpha \otimes x)$ for all $\alpha \in \Omega$ and $x \in A$.*

Proof. This is because f is an Ω-functor if and only if for all $\alpha \in \Omega, x, y \in A$, $\alpha \leq A(x, y)$ implies that $\alpha \leq B(f(x), f(y))$. That means, $\alpha \otimes x \leq y$ implies $\alpha \otimes f(x) \leq f(y)$, which is equivalent to that $f : A_0 \longrightarrow B_0$ preserves order and $\alpha \otimes f(x) \leq f(\alpha \otimes x)$ for all $\alpha \in \Omega$ and $x \in A$. $\Box$

Proposition 3.6 ([35]) *Suppose A and B are tensored Ω-categories, and $f : A \longrightarrow B$ is an Ω-functor. Then, the followings are equivalent:*

(1) *f is a left adjoint.*

(2) *$f : A_0 \longrightarrow B_0$ is a left adjoint and f preserves tensors in the sense that $f(\alpha \otimes x) = \alpha \otimes f(x)$.*

Proof. (1) $\Rightarrow$ (2): We need only show that f preserves tensors. Suppose $g : B \longrightarrow A$ is the right adjoint of f. For all $\alpha \in \Omega, x \in A, y \in B$,

$$\begin{aligned} B(f(\alpha \otimes x), y) &= A(\alpha \otimes x, g(y)) = \alpha \rightarrow A(x, g(y)) \\ &= \alpha \rightarrow B(f(x), y) = B(\alpha \otimes x, y). \end{aligned}$$

Therefore, $f(\alpha \otimes x) = \alpha \otimes f(x)$.

(2) $\Rightarrow$ (1): Suppose $g : B_0 \longrightarrow A_0$ is a right adjoint of $f : A_0 \longrightarrow B_0$. For all $\alpha \in \Omega, x \in A, y \in B$,

$$\begin{aligned} \alpha \leq B(f(x), y) &\iff \alpha \otimes f(x) \leq y \iff f(\alpha \otimes x) \leq y \\ &\iff \alpha \otimes x \leq g(y) \iff \alpha \leq A(x, g(y)). \end{aligned}$$

$\Box$

Similarly, we have the following.

Proposition 3.7 ([35]) *Suppose A and B are cotensored Ω-categories and $f : A \longrightarrow B$ is an Ω-functor. Then, the followings are equivalent:*

(1) *f is a right adjoint.*

(2) *$f : A_0 \longrightarrow B_0$ is a right adjoint and f preserves cotensors in the sense that $f(\alpha \rightarrowtail x) = \alpha \rightarrowtail f(x)$.* $\Box$

Theorem 3.8 ([35]) *An antisymmetric, tensored and cotensored Ω-category A is a cocomplete (hence complete) Ω-lattice if and only if A_0 is a complete lattice.*

Proof. We need only check the sufficiency. Suppose that A_0 is complete. When regarded as a function $A_0 \longrightarrow [A^{\mathrm{op}}, \Omega]_0$, the Yoneda embedding $\mathbf{y}$ is a right adjoint by 3.4(4); and it preserves cotensors by 3.4(1). Thus, $\mathbf{y} : A \longrightarrow [A^{\mathrm{op}}, \Omega]$ has a left adjoint by the above proposition. $\Box$

The following proposition shows that the suprema and infima in a complete Ω-lattice A can be completely described by the lattice structure of A_0 and the tensors and cotensors in A.

Proposition 3.9 *Suppose A is a complete Ω-lattice. We have the following:*

(1) *For every $\lambda \in \Omega^A$, $\sup \lambda = \bigvee_{x\in A}(\lambda(x) \otimes x)$.*

(2) *For every $\mu \in \Omega^A$, $\inf \mu = \bigwedge_{x\in A}(\mu(x) \rightarrowtail x)$.*

Proof. We prove (2) for example. Because the Ω-functor $\inf : [A, \Omega]^{\mathrm{op}} \longrightarrow A$ is, by definition, a right adjoint, it preserves meets and cotensors. And the meets and cotensors in $[A, \Omega]^{\mathrm{op}}$ are exactly the joins and tensors in $[A, \Omega]$ respectively. Therefore,

$$\begin{aligned}
\inf \mu = \inf({\uparrow}\mu) &= \inf\Big(\bigvee_{x\in A}(\mu(x) * \mathbf{y}'(x))\Big) \\
&= \bigwedge_{x\in A} \inf(\mu(x) * \mathbf{y}'(x)) \\
&= \bigwedge_{x\in A}(\mu(x) \rightarrowtail \sup\circ\mathbf{y}(x)) \\
&= \bigwedge_{x\in A}(\mu(x) \rightarrowtail x),
\end{aligned}$$

where the last equality is from that $\sup\circ\mathbf{y} = \mathrm{id}_A$ because $\mathbf{y}$ is injective. □

The above results show that the structure of a complete Ω-lattice A can be completely described by the complete lattice structure of A_0, the tensor $\otimes$ and the cotensor $\rightarrowtail$ on A. By (i), (ii), (iii) and (v) in 3.4, a complete Ω-lattice A is an Ω-module in the category of complete lattices and join-preserving functions [2, 18, 31]. Conversely, given an Ω-module in category of complete lattices and join-preserving functions, i.e., a complete lattice A_0 and binary operation $\otimes : \Omega \times A_0 \longrightarrow A_0$ which satisfies (i), (ii), (iii) and (v) in 3.4, let $A(x, y) = \bigvee\{\alpha \in \Omega \mid \alpha \otimes x \leq y\}$. Then, (1) $A(x, x) \geq I$ for all $x \in A$; (2) for all $x, y, z \in A$,

$$\begin{aligned}
A(x, y) * A(y, z) &= \Big(\bigvee\{\alpha \mid \alpha \otimes x \leq y\}\Big) * \Big(\bigvee\{\beta \mid \beta \otimes y \leq z\}\Big) \\
&= \bigvee\{\alpha * \beta \mid \alpha \otimes x \leq y, \beta \otimes y \leq z\} \\
&\leq \bigvee\{\alpha \mid \alpha \otimes x \leq z\} \\
&= A(x, z).
\end{aligned}$$

Thus, A becomes an Ω-category. Moreover, we say that A is cocomplete. To this end, we show that for all $\lambda : A^{\mathrm{op}} \longrightarrow \Omega$, the supremum of λ in A is given by

$$\sup \lambda = \bigvee_{x\in A}(\lambda(x) \otimes x).$$

By definition of $A(x, y)$, for all $\alpha \in \Omega$, $\alpha \leq A(x, y) \iff \alpha \otimes x \leq y$. Therefore,

$$\alpha \leq A\Big(\bigvee\nolimits_{x\in A}(\lambda(x) \otimes x), y\Big) \iff \bigvee_{x\in A} \alpha \otimes (\lambda(x) \otimes x) \leq y$$

$$
\begin{aligned}
&\iff \bigvee_{x\in A}(\alpha * \lambda(x))\otimes x \leq y\\
&\iff \forall x\in A, \alpha * \lambda(x) \leq A(x,y)\\
&\iff \alpha \leq \bigwedge_{x\in A}(\lambda(x)\to A(x,y)),
\end{aligned}
$$

therefore,

$$A\Big(\bigvee\nolimits_{x\in A}(\lambda(x)\otimes x), y\Big) = \bigwedge_{x\in A}(\lambda(x)\to A(x,y)),$$

which means that $\sup\lambda = \bigvee_{x\in A}(\lambda(x)\otimes x)$.

Therefore, complete Ω-lattices and Ω-modules are essentially the same things. This fact was first pointed out by Stubbe in [35, 37].

Dually, given a complete lattice A_0 and a binary operator $\rightarrowtail: \Omega\otimes A_0^{\mathrm{op}} \longrightarrow A^{\mathrm{op}}$ which satisfies the conditions (i′), (ii′), (iv′) and (v′) in 3.4, let $A(x,y) = \bigvee\{\alpha\in\Omega \mid x\leq \alpha\rightarrowtail y\}$. Then A becomes a complete Ω-lattice.

Examples 3.10 (1) [7, 21] Suppose that A is an Ω-category. Then the Ω-category $[A^{\mathrm{op}},\Omega]$ is tensored and cotensored. Since the underlying poset of $[A^{\mathrm{op}},\Omega]$ is a complete lattice, $[A^{\mathrm{op}},\Omega]$ is a complete Ω-lattice. For any function $G: [A^{\mathrm{op}},\Omega] \longrightarrow \Omega$,

$$\sup\ G = \bigvee_{\phi\in[A^{\mathrm{op}},\Omega]} G(\phi)*\phi,\quad \inf G = \bigwedge_{\phi\in[A^{\mathrm{op}},\Omega]} G(\phi)\to\phi.$$

(2) ([7]) The Ω-category $(\Omega,\to)$ is a complete Ω-lattice since $\Omega\cong[\Omega^{\mathbf{1}}]$, where $\mathbf{1}$ is the singleton discrete Ω-category. Therefore, for all $\mu\in\Omega^\Omega$,

$$\inf\mu = \bigwedge_{y\in\Omega}\mu(y)\to y,\qquad \sup\mu = \bigvee_{y\in\Omega}\mu(y)*y.$$

Theorem 3.11 (Tarski Fixed-point Theorem) *Suppose A is a complete Ω-lattice and $f: A\longrightarrow A$ is an Ω-functor. Then the set of fixed points of f, $\mathbf{Fix}(f) = \{x\in A: f(x) = x\}$, as a subcategory of A, is also a complete Ω-lattice.*

Proof. Firstly, we show that the set of prefixed points of f, $M = \{x\in A: x\leq f(x)\}$, where $\leq$ is the order on A_0, as a subcategory of A, is complete. For any $B\subset M$, $\bigvee B \leq \bigvee_{b\in B} f(b) \leq f(\bigvee B)$, which implies that M is closed under the formation of arbitrary joins in A_0, hence M_0 is a complete lattice. To see that M is a complete Ω-lattice, it is enough to show M is closed under the formation tensors. For all $\alpha\in\Omega$, $x\in M$, $\alpha\otimes x\leq\alpha\otimes f(x)\leq f(\alpha\otimes x)$ and thus $\alpha\otimes x\in M$.

Secondly, note that the image of M under f is also contained in M, thus we can restrict the domain and codomain of f and get a new Ω-functor $f': M\longrightarrow M$, which is also an Ω-functor from M^{op} to M^{op}. The prefixed points of $f': M^{\mathrm{op}}\longrightarrow M^{\mathrm{op}}$ are exactly the fixed points of f. Thus, the Ω-category $\mathbf{Fix}(f)^{\mathrm{op}}$, as a subcategory of M^{op}, is complete and then $\mathbf{Fix}(f)$ is a complete Ω-lattice. $\square$

Proposition 3.12 ([21]) *Suppose B is complete Ω-category. Then for any Ω-category A, the functor category $[A,B]$ is complete.*

Proof. This conclusion has already been proved in [21] by showing that all the weighted limits exist. Here we include another, relatively simpler, proof here.

(1) $[A, B]_0$ is complete. At first, observe that $f \leq g$ in $[A, B]_0$ if and only if $f(x) \leq g(x)$ in B_0 for all $x \in A$. Now suppose $(f_t)_{t \in T}$ is a family of Ω-functors from A to B. Define $f(x) = \bigvee_{t \in T} f_t(x)$ for all $x \in A$, where the join is taken in B_0. It suffices to show that $f \in [A, B]$. Actually, for all $x, y \in A$,

$$\begin{aligned} B(f(x), f(y)) &= B\Big(\bigvee\nolimits_{t \in T} f_t(x), \bigvee\nolimits_{t \in T} f_t(y)\Big) \\ &= \bigwedge_{t \in T} B\Big(f_t(x), \bigvee\nolimits_{t \in T} f_t(y)\Big) \\ &\geq \bigwedge_{t \in T} B(f_t(x), f_t(y)) \\ &\geq A(x, y). \end{aligned}$$

(2) Denote the tensor in B by $\otimes_B$. For each $\alpha \in \Omega$, define a function $\alpha \otimes_B (\_) : B \longrightarrow B$ by $\alpha \otimes_B (\_)(x) = \alpha \otimes_B x$. Then $\alpha \otimes_B (\_)$ is an Ω-functor since if $\beta \leq B(x, y)$ then $\beta \otimes_B x \leq y$, and hence $\beta \otimes_B (\alpha \otimes_B x) = \alpha \otimes_B (\beta \otimes_B x) \leq \alpha \otimes_B y$. Thus, $\beta \leq B(\alpha \otimes_B x, \alpha \otimes_B y)$.

(3) Given $\alpha \in \Omega, f \in [A, B]$, let $(\alpha \otimes f)(x) = \alpha \otimes_B f(x)$. Then $\alpha \otimes f \in [A, B]$ since it is the composition of f and $\alpha \otimes_B (\_)$. We leave it to the reader to check that $\otimes : \Omega \times [A, B]_0 \longrightarrow [A, B]_0$ satisfies the conditions (i), (ii), (iii) and (v) in 3.4, hence $[A, B]$ is a complete Ω-lattice. □

Definition 3.13 A Ω-functor $f : A \longrightarrow B$ between the complete Ω-lattices is called a complete Ω-lattice morphism if it has both left and right adjoints.

Complete Ω-lattices and complete Ω-lattice morphisms form a category, which shall be denoted Ω-**CLat**.

Proposition 3.14 ([21]) *Suppose $\{A_i : i \in J\}$ is a family of complete Ω-lattices. The product $\prod_{i \in J} A_i$ is also complete.*

Proof. Clearly, $\prod_{i \in J} (A_i)_0$ is a complete lattice. We define an action of Ω on $\prod_{i \in J} (A_i)_0$ by $\alpha \otimes (a_i)_{i \in J} = (\alpha \otimes_i a_i)_{i \in J}$, $\alpha \in \Omega$, $(a_i)_{i \in J} \in \prod_{i \in J} (A_i)_0$, where $\otimes_i$ is the tensor on A_i. It is easy to check that $\otimes$ satisfies the conditions (i), (ii), (iii) and (v) in 3.4, so it determines an Ω-categorical structure on $\prod_{i \in J} (A_i)_0$. What remains is to show that $\prod_{i \in J} (A_i)_0$ together with this Ω-categorical structure coincides with the product $\prod_{i \in J} A_i$. In fact, for any $\alpha \in \Omega$ and $a, b \in \prod_{i \in J} A_i$,

$$\begin{aligned} \alpha \otimes a \leq b &\iff \forall i \in J, \alpha \otimes_i a_i \leq b_i \\ &\iff \forall i \in J, \alpha \leq A_i(a_i, b_i) \\ &\iff \alpha \leq \bigwedge_{i \in J} A_i(a_i, b_i) = \prod_{i \in J} A_i(a, b). \end{aligned}$$

□

For each $j \in J$, the projection $p_j : \prod_{i\in J} A_i \longrightarrow A_j$ is a complete Ω-lattice morphism. The left adjoint is given by

$$q_j : A_j \cong A_j \times \prod_{i\neq j}\{0_{A_i}\} \hookrightarrow \prod_{i\in J} A_i,$$

and the right adjoint is given by

$$q'_j : A_j \cong A_j \times \prod_{i\neq j}\{1_{A_i}\} \hookrightarrow \prod_{i\in J} A_i.$$

Thus, $\prod_{i\in J} A_i$ is the product of $\{A_i : i \in J\}$ in the category Ω-**CLat**.

Proposition 3.15 *Suppose f and g are complete Ω-lattice morphisms from a complete Ω-lattice A to a complete Ω-lattice B. Then the equalizer of f and g exists in the category Ω-**CLat**.*

Proof. Let $E = \{x \in A : f(x) = g(x)\}$ and $E(x, y) = A(x, y)$. Then E is a subcategory of A. It is enough to show that E is complete and the embedding $i : E \longrightarrow A$ is a complete Ω-lattice morphism. Take any Ω-subset $\mu : E \longrightarrow \Omega$. We have that $f(\sup_A i(\mu)) = \sup_B f \circ i(\mu) = \sup_B g \circ i(\mu) = g(\sup_A i(\mu))$ because f and g preserves sups. Thus $\sup_A i(\mu) \in E$ and then $\sup_E \mu = \sup_A i(\mu)$. Similarly, we can check that $\inf_E \mu = \inf_A i(\mu)$. Therefor, E is complete and i is a complete Ω-lattice morphism indeed. $\Box$

Recall that a category is complete if and only if the products and the equalizers exist. Clearly, Ω-**CLat** is a complete category.

4 Completely distributive Ω-lattices

Definition 4.1 [37, 42] A complete Ω-lattice A is said to be completely distributive if the functor $\sup : [A^{\mathrm{op}}, \Omega] \longrightarrow A$ has a left adjoint, denoted, $\Downarrow : A \longrightarrow [A^{\mathrm{op}}, \Omega]$.

Clearly, when $\Omega = \mathbf{2}$, completely distributive Ω-lattices coincide with the constructive completely distributive lattices in [11, 28, 29, 40]

Example 4.2 The Ω-category $(\Omega, \rightarrow)$ is completely distributive, i.e., the Ω-functor $\sup : [\Omega^{\mathrm{op}}, \Omega] \longrightarrow \Omega$ has a left adjoint. This is a special case of the general result Proposition 4.5 below. However, we shall construct here a left adjoint of $\sup : [\Omega^{\mathrm{op}}, \Omega] \longrightarrow \Omega$ explicitly. At first, for each $\phi \in [\Omega^{\mathrm{op}}, \Omega]$, we have:

(1) $\phi : \Omega \longrightarrow \Omega$ is a decreasing function;

(2) For all $x \in \Omega, x * \phi(x) \leq \phi(I)$ since $x = I \rightarrow x \leq \phi(x) \rightarrow \phi(I)$;

(3) For all $x \in \Omega$, $\phi(I) \leq (x \rightarrow I) \rightarrow \phi(x)$ since $x \rightarrow I \leq \phi(I) \rightarrow \phi(x)$.

Thus, for each $\phi \in [\Omega^{\mathrm{op}}, \Omega]$,

$$\sup \phi = \bigvee_{x\in\Omega} x * \phi(x) = \phi(I) \quad \text{and} \quad \bigwedge_{x\in\Omega}((x \rightarrow I) \rightarrow \phi(x)) = \phi(I).$$

For each $x \in \Omega$, let $\Downarrow(x) : \Omega \longrightarrow \Omega$ be given by $\Downarrow(x)(t) = x * (t \to I)$ for all $t \in \Omega$. Clearly $\Downarrow(x) \in [\Omega^{\mathrm{op}}, \Omega]$ and we claim that $\Downarrow : \Omega \longrightarrow [\Omega^{\mathrm{op}}, \Omega]$ is a left adjoint of $\sup : [\Omega^{\mathrm{op}}, \Omega] \longrightarrow \Omega$. In fact, for all $\phi \in [\Omega^{\mathrm{op}}, \Omega]$ and $x \in \Omega$,

$$\begin{aligned} [\Omega^{\mathrm{op}}, \Omega](\Downarrow(x), \phi) &= \bigwedge_{t\in\Omega}((x * (t \to I)) \to \phi(t)) = \bigwedge_{t\in\Omega} x \to ((t \to I) \to \phi(t)) \\ &= x \to \bigwedge_{t\in\Omega}(t \to I) \to \phi(t) = x \to \phi(I) \\ &= \Omega(x, \sup\phi). \end{aligned}$$

Suppose A is a completely distributive Ω-lattice. It is easily seen that for all $a \in A$ and $\lambda \in [A^{\mathrm{op}}, \Omega]$, $\sup(\Downarrow(a)) = a$ and $\sup\lambda \geq a$ if and only if $\Downarrow(a) \leq \lambda$. And, following the terminologies in the series [11, 28, 29, 40] on constructive completely distributive lattices, we call $\Downarrow(a)(x)$ the degree that x is *totally below* a.

Proposition 4.3 (Also in [37]) *Suppose A is a completely distributive Ω-lattice. Then the totally below relation on A is interpolative in the sense that for all $x, y \in A$,*

$$\Downarrow(x)(y) = \bigvee_{z\in A}\Big(\Downarrow(x)(z) * \Downarrow(z)(y)\Big).$$

Proof. Let $\lambda(w) = \bigvee_{z\in A}\Big(\Downarrow(x)(z) * \Downarrow(z)(w)\Big)$ for all $w \in A$. Then $\lambda \in [A^{\mathrm{op}}, \Omega]$. Because

$$\begin{aligned} \sup\lambda &= \sup\bigvee_{z\in A}\Big(\Downarrow(x)(z) * \Downarrow(z)\Big) \\ &= \sup\bigvee_{w\in A}\Big(\Downarrow(x)(z) * \Downarrow(z)(w)\Big) \otimes w \\ &= \bigvee_{w\in A}\bigvee_{z\in A}\Big((\Downarrow(x)(z) * \Downarrow(z)(w)) \otimes w\Big) \\ &= \bigvee_{z\in A}\Big[\Downarrow(x)(z) \otimes \Big(\bigvee\nolimits_{w\in A}(\Downarrow(z)(w) \otimes w\Big)\Big] \\ &= \bigvee_{z\in A}\Big[\Downarrow(x)(z) \otimes \sup(\Downarrow(z))\Big] \\ &= \bigvee_{z\in A}\Big(\Downarrow(x)(z) \otimes z\Big) \\ &= \sup(\Downarrow(x)) = x. \end{aligned}$$

Therefore $\lambda \geq \Downarrow(x)$ and particularly,

$$\Downarrow(x)(y) \leq \bigvee_{z\in A}\Big(\Downarrow(x)(z) * \Downarrow(z)(y)\Big).$$

Conversely, since $\Downarrow : A \longrightarrow [A^{\mathrm{op}}, \Omega]$ is an Ω-functor, for each $z \in A$, we have that

$$\Downarrow(x)(z) \leq A(z, x) \leq \bigwedge_{w\in A}\Big(\Downarrow(z)(w) \to \Downarrow(x)(w)\Big) \leq \Downarrow(z)(y) \to \Downarrow(x)(y).$$

Consequently,

$$\Downarrow(x)(y) \geq \Downarrow(x)(z) * \Downarrow(z)(y)$$

for all $z \in A$, and thus,

$$\Downarrow(x)(y) \geq \bigvee_{z \in A} \Big(\Downarrow(x)(z) * \Downarrow(z)(y) \Big).$$

□

The category of all the completely distributive Ω-lattices and the complete Ω-lattice morphisms is denoted Ω-**CD**, which is a full subcategory of Ω-**CLat**.

Example 4.4 Both the singleton discrete Ω-category **1** and the terminal Ω-category $\top$ are completely distributive Ω-lattices.

Proposition 4.5 (Also in [37]) *Suppose A is an Ω-category. Then $[A^{\rm op}, \Omega]$ is a completely distributive Ω-lattice. Particularly, both $[\Omega^A] = [|A|, \Omega]$ and $\Omega \cong [\Omega^{\mathbf{1}}]$ are completely distributive Ω-lattices.*

Proof. Let $\mathbf{y}_A : A \longrightarrow [A^{\rm op}, \Omega]$ be the Yoneda embedding. By 2.10, the Ω-functor $\mathbf{y}_A^{\leftarrow} : [[A^{\rm op}, \Omega]^{\rm op}, \Omega] \longrightarrow [A^{\rm op}, \Omega]$ has a left adjoint. Thus, it suffices to show that the Ω-functors $\sup : [[A^{\rm op}, \Omega]^{\rm op}, \Omega] \longrightarrow [A^{\rm op}, \Omega]$ and $\mathbf{y}_A^{\leftarrow}$ coincide with each other, i.e. $\sup \Phi = \mathbf{y}_A^{\leftarrow}(\Phi) = \Phi \circ \mathbf{y}_A$ for all Ω-functor $\Phi : [A^{\rm op}, \Omega]^{\rm op} \longrightarrow \Omega$. Indeed, for any $x \in A$,

$$\sup \Phi(x) = \bigvee_{\phi \in [A^{\rm op}, \Omega]} \Phi(\phi) * \phi(x) \geq \Phi(\mathbf{y}_A(x)) * \mathbf{y}_A(x)(x) \geq \Phi(\mathbf{y}_A(x)).$$

On the other hand, because Φ is an Ω-functor from $[A^{\rm op}, \Omega]^{\rm op}$ to Ω,

$$\sup \Phi(x) = \bigvee_{\phi \in [A^{\rm op}, \Omega]} \Phi(\phi) * \phi(x) = \bigvee_{\phi \in [A^{\rm op}, \Omega]} \Phi(\phi) * [A^{\rm op}, \Omega](\mathbf{y}_A(x), \phi) \leq \Phi(\mathbf{y}_A(x)).$$

Therefore, $\sup(\Phi) = \Phi \circ \mathbf{y}_A$. □

Theorem 4.6 *Suppose $\{A_i : i \in J\}$ is a family of completely distributive Ω-lattices. The product $A = \prod_{i \in J} A_i$ is also a completely distributive Ω-lattice.*

Proof. Step 1. We show that for all $\phi \in [A^{\rm op}, \Omega]$, and $j \in J$, the image of ϕ under the projection $p_j : \prod_{i \in J} A_i \longrightarrow A_j$ is given by $p_j(\phi)(t) = \phi(q_j(t))$ for all $t \in A_j$, where $q_j : A_j \cong A_j \times \prod_{i \neq j} \{0_{A_i}\} \longrightarrow \prod_{i \in J} A_i$ is the left adjoint of the projection $p_j : \prod_{i \in J} A_i \longrightarrow A_j$. In fact, since ϕ is a decreasing function from the complete lattice A_0 to the complete lattice Ω, for all $t \in A_j$ and $x \in p^{-1}(t)$, we have that $\phi(q_j(t)) \geq \phi(x)$. Therefore,

$$p_j(\phi)(t) = \bigvee_{x \in p_j^{-1}(t)} \phi(x) = \phi(q_j(t)).$$

Step 2. We construct a left adjoint of the Ω-functor $\sup : [A^{\rm op}, \Omega] \longrightarrow A$. For each $j \in J$, let $d_j : A \longrightarrow \Omega$ be given by

$$d_j(a)(x) = \begin{cases} \Downarrow_j (a_j)(t), & x = q_j(t) \text{ for some } t \in A_j; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\Downarrow(a) = \downarrow(\bigvee_{j\in J} d_j(a))$. Then we claim that $\Downarrow: A \longrightarrow [A^{\rm op}, \Omega]$ is a left adjoint of $\sup : [A^{\rm op}, \Omega] \longrightarrow A$. Actually, for all $a \in A$ and $\phi \in [A^{\rm op}, \Omega]$,

$$\begin{aligned}
[A^{\rm op}, \Omega](\Downarrow(a), \phi) &= [A^{\rm op}, \Omega]\Big(\downarrow\Big(\bigvee\nolimits_{j\in J} d_j(a)\Big), \phi\Big) \\
&= [\Omega^A]\Big(\bigvee\nolimits_{j\in J} d_j(a), \phi\Big) \quad \text{(Example 2.11)} \\
&= \bigwedge_{j\in J}\bigwedge_{x\in A} d_j(a)(x) \to \phi(x) \\
&= \bigwedge_{j\in J}\bigwedge_{t\in A_j} \Downarrow_j (a_j)(t) \to p_j(\phi)(t) \\
&= \bigwedge_{j\in J} [A_j^{\rm op}, \Omega](\Downarrow_j (a_j), p_j(\phi)) \\
&= \bigwedge_{j\in J} A_j(a_j, \sup{}_{A_j} p_j(\phi)) \\
&= A(a, \sup \phi),
\end{aligned}$$

where the last equality holds because the complete Ω-lattice morphism p_j preserves sups, i.e. $\sup_{A_j} p_j(\phi) = p_j(\sup_A \phi)$. Therefor, $\Downarrow$ is a left adjoint of $\sup : [A^{\rm op}, \Omega] \longrightarrow A$ as claimed. $\Box$

5 Subalgebras and quotient algebras

Definition 5.1 Suppose that A is an Ω-category and B is a subcategory of A. Then B is said to be a subalgebra of A if the embedding functor $i : B \longrightarrow A$ preserves both sups and infs.

Suppose A is a complete Ω-lattice and B is a subcategory of A. It is routine to check that the following conditions are equivalent: (1) B is a subcategory of A; (2) B is closed with respect to tensors and cotensors in A and B_0 is closed with respect to joins and meets in A_0; (3) the embedding functor $i : B \longrightarrow A$ has both a left and a right adjoint.

By (2), every subalgebra B of a complete Ω-lattice A is itself a complete Ω-lattice. Here is another proof of this fact. Let $i : B \longrightarrow A$ denote the embedding and $k : A \longrightarrow B$ be a left adjoint of i. Then the Yoneda embedding $\mathbf{y}_B : B \longrightarrow [B^{\rm op}, \Omega]$ can be written as a composition $i^{\leftarrow} \circ \mathbf{y}_A \circ i : B \longrightarrow A \longrightarrow [A^{\rm op}, \Omega] \longrightarrow [B^{\rm op}, \Omega]$. Thus, $\mathbf{y}_B$ has a left adjoint given by $\sup_B = k \circ \sup_A \circ i_\ell^{\rightarrow} : [B^{\rm op}, \Omega] \longrightarrow [A^{\rm op}, \Omega] \longrightarrow A \longrightarrow B$.

Moreover, for completely distributive Ω-lattices, we have the following.

Proposition 5.2 *Suppose L is a completely distributive Ω-lattice and M is a subalgebra of L. Then M is also completely distributive.*

Proof. Suppose that $i : M \longrightarrow L$ is the corresponding embedding and $k : L \longrightarrow M$ is a left adjoint of i. Let $\Downarrow_L : L \longrightarrow [L^{\mathrm{op}}, \Omega]$ be a left adjoint of $\sup_L : [L^{\mathrm{op}}, \Omega] \longrightarrow L$. Then, we say that $\Downarrow_M = k_\ell^{\rightarrow} \circ \Downarrow_L \circ i : M \longrightarrow L \longrightarrow [L^{\mathrm{op}}, \Omega] \longrightarrow [M^{\mathrm{op}}, \Omega]$ is a left adjoint of $\sup_M : [M^{\mathrm{op}}, \Omega] \longrightarrow \Omega$, hence, M is completely distributive.

Indeed, for all $x \in A$ and $\phi \in [M^{\mathrm{op}}, \Omega]$,

$$[M^{\mathrm{op}}, \Omega](\Downarrow_M (x), \phi] = [M^{\mathrm{op}}, \Omega](k_\ell^{\rightarrow} \circ \Downarrow_L \circ i(x), \phi) = L(i(x), \sup{}_L \circ k^{\leftarrow}(\phi)).$$

And

$$M(x, \sup{}_M(\phi)) = L(i(x), i(\sup{}_M \phi)) = L(i(x), \sup{}_L \circ i_\ell^{\rightarrow}(\phi))$$

since $i : M \longrightarrow L$ preserves sups.

So, it suffices to show that $k^{\leftarrow}(\phi) = i_\ell^{\rightarrow}(\phi)$. In fact, for all $y \in A$,

$$\begin{aligned} i_\ell^{\rightarrow}(\phi)(y) &= \bigvee_{x \in M} \phi(x) * L(y, i(x)) = \bigvee_{x \in M} \phi(x) * M(k(y), x) \\ &\geq \phi(k(y)) * M(k(y), k(y)) \geq \phi(k(y)) = k^{\leftarrow}(\phi)(y). \end{aligned}$$

On the other hand, since $\phi : M^{\mathrm{op}} \longrightarrow \Omega$ is an Ω-functor,

$$\begin{aligned} i_\ell^{\rightarrow}(\phi)(y) &= \bigvee_{x \in M} \phi(x) * L(y, i(x)) = \bigvee_{x \in M} \phi(x) * M(k(y), x) \\ &\leq \phi(k(y)) = k^{\leftarrow}(\phi)(y). \end{aligned}$$

Thus, $k^{\leftarrow}(\phi) = i_\ell^{\rightarrow}(\phi)$ as desired. $\square$

Corollary 5.3 Ω-**CD** *is a complete category.*

Proof. This is because that Ω-**CD** has a terminal object and products by 4.4 and 4.6, and that it has equalizers by 5.2 and 3.15. $\square$

In the following we shall show that the subalgebras of a completely distributive Ω-lattice A can be equivalently described by certain closure operators on A.

Definition 5.4 A closure operator on an Ω-category A is an idempotent Ω-functor $c : A \longrightarrow A$ such that $A(x, c(x)) \geq I$ for all $x \in A$.

Lemma 5.5 *Suppose A is a complete Ω-lattice and $B \subseteq A$ is a subalgebra of A. Let $c : A \longrightarrow A$ be given by $c(x) = \bigwedge\{y \in B \mid x \leq y\}$. Then c is a cocontinuous closure operator on A.*

Proof. (1) c is an Ω-functor. It suffices to show that $\alpha \otimes c(x) \leq c(\alpha \otimes x)$ for all $\alpha \in \Omega$ and $x \in A$. Indeed, for all $z \in B$, if $\alpha \otimes x \leq z$, then $x \leq \alpha \rightarrowtail z$ by **??**(1). Since B is a subalgebra of A, $\alpha \rightarrowtail z \in B$. Hence, $c(x) \leq \alpha \rightarrowtail z$ and $\alpha \otimes c(x) \leq c(\alpha \otimes x)$ by arbitrariness of z.

(2) For every $x \in A$, $A(x, c(x)) \geq I$. This is trivial by definition.

(3) $c : A_0 \longrightarrow A_0$ preserves joins. Suppose that $(x_t)_{t\in T} \subseteq A$. Then

$$c\Big(\bigvee_{t\in T} x_t\Big) = c \circ c\Big(\bigvee_{t\in T} x_t\Big) \geq c\Big(\bigvee_{t\in T} c(x_t)\Big) \geq \bigvee_{t\in T} c(x_t).$$

On the other hand, since $\bigvee_{t\in T} x_t \leq \bigvee_{t\in T} c(x_t)$ and $\bigvee_{t\in T} c(x_t) \in B$ by assumption, we have that $c\Big(\bigvee_{t\in T} x_t\Big) \leq \bigvee_{t\in T} c(x_t)$. Hence $c\Big(\bigvee_{t\in T} x_t\Big) = \bigvee_{t\in T} c(x_t)$.

(4) c preserves tensors, i.e., $c(\alpha \otimes x) = \alpha \otimes c(x)$ for all $\alpha \in \Omega$ and $x \in A$. By (1), we need only check that $c(\alpha \otimes x) \leq \alpha \otimes c(x)$. By definition, $c(x)$ is the least element in B such that $x \leq c(x)$. Thus, $\alpha \otimes x \leq \alpha \otimes c(x)$. Consequently, $c(\alpha \otimes x) \leq \alpha \otimes c(x)$ since $\alpha \otimes c(x) \in B$.

Therefore, c is a cocontinuous closure operator. $\Box$

Lemma 5.6 *Suppose that A is a complete Ω-lattice and $c : A \longrightarrow A$ is a cocontinuous closure operator. Then $c(A) = \{c(x) \mid x \in A\}$ is a subalgebra of A.*

Proof. (1) Suppose $(x_t)_{t\in T} \subseteq c(A)$. Then

$$c\Big(\bigvee_{t\in T} x_t\Big) = \bigvee_{t\in T} c(x_t) = \bigvee_{t\in T} x_t$$

and

$$\bigwedge_{t\in T} x_t \leq c\Big(\bigwedge_{t\in T} x_t\Big) \leq \bigwedge_{t\in T} c(x_t) = \bigwedge_{t\in T} x_t.$$

Thus, $(c(A))_0$ is a complete sublattice of A_0.

(2) For all $\alpha \in \Omega$ and $x \in c(A)$, $\alpha \otimes x \in c(A)$. This is because

$$\alpha \otimes x \leq c(\alpha \otimes x) = \alpha \otimes c(x) = \alpha \otimes x.$$

(3) For all $\alpha \in \Omega$ and $x \in c(A)$, $\alpha \rightarrowtail x \in c(A)$, i.e., $c(\alpha \rightarrowtail x) = \alpha \rightarrowtail x$. Since

$$A(c(y), x) \leq A(y, x) \leq A(c(y), c(x)) = A(c(y), x)$$

for all $y \in A$, we have that

$$\begin{aligned}
\alpha \rightarrowtail x &= \bigvee \{y \in A \mid \alpha \leq A(y, x)\} \\
&= \bigvee \{y \in A \mid \alpha \leq A(c(y), x)\} \\
&= \bigvee \{c(y) \mid y \in A \text{ and } \alpha \leq A(y, x)\} \\
&= c(\alpha \rightarrowtail x).
\end{aligned}$$

Therefore, $c(A)$ is a subalgebra of A. $\Box$

A combination of the above two lemmas yields the following.

Proposition 5.7 *The subalgebras of a completely distributive Ω-lattice A correspond bijectively to the cocontinuous closure operators on A.* $\Box$

Definition 5.8 Let A and B be Ω-categories. B is said to be a quotient algebra of A if there is a surjective Ω-functor $f : A \longrightarrow B$ such that f has both a left adjoint and a right adjoint.

Suppose that A is a complete Ω-lattice and B is a quotient algebra of A. By definition, there is a surjective functor $f : A \longrightarrow B$ which has both a left adjoint and a right adjoint. Let $k : L \longrightarrow M$ be a right adjoint of f. Then k is an isometry and the Yoneda embedding $\mathbf{y}_B : B \longrightarrow [B^{\rm op}, \Omega]$ can be written as a composition $k^{\leftarrow} \circ \mathbf{y}_A \circ k : B \longrightarrow A \longrightarrow [A^{\rm op}, \Omega] \longrightarrow [B^{\rm op}, \Omega]$. Thus, $\mathbf{y}_B$ has a left adjoint given by $\sup_B = f \circ \sup_A \circ f^{\leftarrow} : [B^{\rm op}, \Omega] \longrightarrow [A^{\rm op}, \Omega] \longrightarrow A \longrightarrow B$. Therefore, B is a complete Ω-lattice. To conclude, every quotient algebra of a complete Ω-lattice is also a complete Ω-lattice.

Proposition 5.9 *Every quotient algebra of a completely distributive Ω-lattice is also a completely distributive Ω-lattice.*

Proof. Suppose that L is a completely distributive Ω-lattice and $f : L \longrightarrow M$ is a surjective Ω-functor with a left adjoint $j : M \longrightarrow L$ and a right adjoint $k : M \longrightarrow L$. Then, M is a complete Ω-lattice by the above observation and j, k are injective isometric Ω-functors by 2.9. Let $\Downarrow_M = f_\ell^{\rightarrow} \circ \Downarrow_L \circ j$, where $\Downarrow_L$ is the left adjoint of $\sup_L$. We left it to the reader to check that $\Downarrow_M$ is a left adjoint of $\sup_M$. Hence, M is completely distributive. $\Box$

The following classical characterization of complete distributivity was established by Raney and Büchi independently in [8, 26, 27].

Theorem 5.10 (Raney-Büchi) *A complete Ω-lattice L is completely distributive if and only if there is a set X such that L is a quotient algebra of some subalgebra of $[\Omega^X]$.*

Proof. Sufficiency: This follows from 4.5, 5.2 and 5.9 immediately.

Necessity: since $[L^{\rm op}, \Omega]$ is a subcategory of $[\Omega^L]$ and the inclusion $i : [L^{\rm op}, \Omega] \longrightarrow [\Omega^L]$ has both a left adjoint and a right adjoint, $[L^{\rm op}, \Omega]$ is a subalgebra of $[\Omega^L]$. Since $\sup : [L^{\rm op}, \Omega] \longrightarrow L$ is surjective and has both a left adjoint $\Downarrow : L \longrightarrow [L^{\rm op}, \Omega]$ and a right adjoint $\mathbf{y} : L \longrightarrow [L^{\rm op}, \Omega]$, L is a quotient algebra of $[L^{\rm op}, \Omega]$. $\Box$

Definition 5.11 A kernel operator on an Ω-category A is an idempotent Ω-functor $k : A \longrightarrow A$ such that $A(k(x), x) \geq I$ for all $x \in A$.

Let A be a complete Ω-lattice and $k : A \longrightarrow A$ be a cocontinuous kernel operator on A. Define $x \sim y$ in A if $k(x) = k(y)$. Then

(1) If $x_t \sim y_t$ for all $t \in T$, then $\bigvee_{t\in T} x_t \sim \bigvee_{t\in T} y_t$. This is trivial since k preserves joins.

(2) If $x \sim y$, then $\alpha \otimes x \sim \alpha \otimes y$ for all $\alpha \in \Omega$ because k preserves tensors.

(3) If $x_t \sim y_t$ for all $t \in T$, then $\bigwedge_{t\in T} x_t \sim \bigwedge_{t\in T} y_t$. To see this, note at first that

$$k\Big(\bigwedge\nolimits_{t\in T} x_t\Big) \leq \bigwedge\nolimits_{t\in T} k(x_t) = \bigwedge\nolimits_{t\in T} k(y_t) \leq \bigwedge\nolimits_{t\in T} y_t.$$

So

$$k\Big(\bigwedge\nolimits_{t\in T} x_t\Big) = k\circ k\Big(\bigwedge\nolimits_{t\in T} x_t\Big) \le k\Big(\bigwedge\nolimits_{t\in T} y_t\Big).$$

Exchanging the roles of x and y we obtain that

$$k\Big(\bigwedge\nolimits_{t\in T} y_t\Big) \le k\Big(\bigwedge\nolimits_{t\in T} x_t\Big).$$

Therefore,

$$k\Big(\bigwedge\nolimits_{t\in T} y_t\Big) = k\Big(\bigwedge\nolimits_{t\in T} x_t\Big).$$

(4) If $x \sim y$, then, for all $\alpha \in \Omega$, $\alpha \rightarrowtail x \sim \alpha \rightarrowtail y$. That is, $k(\alpha \rightarrowtail x) = k(\alpha \rightarrowtail y)$.

To see this, we assert at first that for all $\alpha \in \Omega$ and $x \in A$, the set $\{k(z) \in A \mid \alpha \le A(z,x)\}$ equals the set $\{k(z) \in A \mid \alpha \le A(k(z), k(x))\}$.

The inclusion $\{k(z) \in A \mid \alpha \le A(z,x)\} \subseteq \{k(z) \in A \mid \alpha \le A(k(z), k(x))\}$ is trivial since k is an Ω-functor. For the converse inclusion, suppose that $\alpha \le A(k(z), k(x))$. Then $\alpha \le A(k(z), k(x)) * A(k(x), x) \le A(k(z), x)$. Thus, $k(z) = k(k(z)) \in \{k(z) \in A \mid \alpha \le A(z,x)\}$.

Therefore,

$$\begin{aligned}
k(\alpha \rightarrowtail x) &= k\Big(\bigvee\{z \in A \mid \alpha \le A(z,x)\}\Big)\\
&= \bigvee\{k(z) \in A \mid \alpha \le A(z,x)\}\\
&= \bigvee\{k(z) \in A \mid \alpha \le A(k(z),k(x))\}\\
&= \bigvee\{k(z) \in A \mid \alpha \le A(k(z),k(y))\}\\
&= \bigvee\{k(z) \in A \mid \alpha \le A(z,y)\}\\
&= k\Big(\bigvee\{z \in A \mid \alpha \le A(z,y)\}\Big)\\
&= k(\alpha \rightarrowtail y).
\end{aligned}$$

By (1)-(4) in the above we see that the subset $R = \{(x,y) \mid x \sim y\} \subseteq A \times A$ is not only an equivalence relation on A, but also a subalgebra of $A \times A$. Let $B = A/\sim$ and $q : A \longrightarrow B$ be the corresponding quotient map. We define an Ω-category structure on B as follows: for all $[x], [y] \in B$, let $B([x],[y]) = A(k(x), k(y))$. Then it is easy to verify that B becomes an Ω-category and $q : A \longrightarrow B$ is an Ω-functor. An we leave it to the reader to check that B is a complete Ω-lattice and that q preserves joins, tensors, meets, and cotensors. Thus, q has, at the same time, a left adjoint and a right adjoint. That means, B is a quotient algebra of A. Therefore, every cocontinuous kernel operator on a complete Ω-lattice A is associated with a quotient algebra of A.

On the other hand, suppose that B is a quotient algebra of a complete Ω-lattice A with the quotient map $q : A \longrightarrow B$. By definition, q has a left adjoint $f : B \longrightarrow A$. Let $k = f \circ q : A \longrightarrow B \longrightarrow A$. Then k is a cocontinuous kernel operator on A.

We leave it to the reader to check that the above processes from quotient algebras of a complete Ω-lattice A to the cocontinuous kernel operators on A and vice versa are inverse to each other. Particularly, we arrive at the following.

Proposition 5.12 *The quotient algebras of a completely distributive Ω-lattice A correspond bijectively to the cocontinuous kernel operators on A.* $\square$

For each complete Ω-lattice A, A_0 must be a complete lattice. But, the complete distributivity of A (as an Ω-lattice) does not imply the complete distributivity of A_0. However, we have the following.

Proposition 5.13 *The following two conditions are equivalent:*

(1) *The complete lattice Ω is completely distributive.*

(2) *For every completely distributive Ω-lattice A, A_0 is completely distributive.*

Proof. (2) $\Rightarrow$ (1): Since $(\Omega, \rightarrow)$ is a completely distributive Ω-lattice, thus, $\Omega = (\Omega, \rightarrow)_0$ is a completely distributive lattice.

(1) $\Rightarrow$ (2): Suppose Ω is a completely distributive complete lattice. Then, $[\Omega^A]_0$ is a completely distributive complete lattice. Since $\sup : [A^{\mathrm{op}}, \Omega] \longrightarrow A$ preserves sups and infs, A is a quotient algebra of $[A^{\mathrm{op}}, \Omega]$, which is a subalgebra of $[\Omega^A]$. Therefore, A_0 is a quotient algebra of a subalgebra $[A^{\mathrm{op}}, \Omega]_0$ of $[\Omega^X]_0$. Letting $\Omega = \mathbf{2}$ in 5.10, we obtain that A_0 is a completely distributive lattice. $\square$

6 The Ω-category of left adjoints

In this section, we discuss the complete distributivity of functor categories. The main result is that if A and B are completely distributive Ω-lattices, then so is the subcategory of the functor category $[A, B]$ consisting of left adjoints from A to B.

Suppose A and B are Ω-categories. Let $[A \rightarrow_\ell B]$ denote the subcategory of $[A, B]$ consisting of left adjoint functors, i.e., every element in $[A \rightarrow_\ell B]$ has a right adjoint.

Proposition 6.1 *Suppose A and B are complete Ω-lattices. Then $[A \rightarrow_\ell B]$ is a complete Ω-lattice.*

Proof. (1) Suppose that $(f_t)_{t\in T} \subseteq [A \rightarrow_\ell B]$, the pointwise join $f = \bigvee_{t\in T} f_t$ given by $f(x) = \bigvee_{t\in T} f_t(x)$ is a cocontinuous Ω-functor $A \longrightarrow B$, where the join is taken in the complete lattice B_0. This is because that (i)

$$f(\alpha \otimes x) = \bigvee_{t\in T} f_t(\alpha \otimes x) = \bigvee_{t\in T} \alpha \otimes f_t(x) = \alpha \otimes \bigvee_{t\in T} f_t(x) = \alpha \otimes f(x),$$

i.e., f preserves tensors; and (ii)

$$f(\bigvee\nolimits_{i\in J} x_i) = \bigvee_{t\in T} f_t(\bigvee\nolimits_{i\in J} x_i) = \bigvee_{i\in J}\bigvee_{t\in T} f_t(x_i) = \bigvee_{i\in J} f(x_i),$$

i.e., $f : A_0 \longrightarrow B_0$ preserves joins. So, $[A \rightarrow_\ell B]_0$ is a complete lattice.

(2) For every $f \in [A \rightarrow_\ell B]$ and $\alpha \in \Omega$, the tensor $\alpha \otimes f$ of α and f in $[A, B]$ is cocontinuous, i.e., $[A \rightarrow_\ell B]$ is closed under tensor in $[A, B]$. The proof is trivial and hence omitted here.

Therefore, $[A \rightarrow_\ell B]$ is a complete Ω-lattice with the tensor inherited from $[A, B]$. $\square$

It should be noted that though $[A \rightarrow_\ell B]$ is a complete Ω-lattice, it is not necessarily a subalgebra of $[A, B]$.

Lemma 6.2 *Suppose A is an Ω-category and B is a completely distributive Ω-lattice. Then $[A, B]$ is also a completely distributive Ω-lattice.*

Proof. Indeed, $[A, B]$ is a subalgebra of the completely distributive Ω-lattice $[|A|, B] = B^{|A|}$. To this end, we need only check that (1) for all $\alpha \in \Omega$ and $f \in [A, B]$, the tensor $\alpha \otimes f$ and cotensor $\alpha \rightarrowtail f$ in $B^{|A|}$ are Ω-functors; and (2) for every family $(f_t)_{t\in T} \subseteq [A, B]$, the pointwise join $\bigvee_{t\in T} f_t$ and pointwise meet $\bigwedge_{t\in T} f_t$ are Ω-functors. The details are left to the reader. $\square$

Theorem 6.3 *Suppose A and B are completely distributive Ω-lattices. Then Ω-category $[A \rightarrow_\ell B]$ of left adjoints is a completely distributive Ω-lattice.*

Proof. Our strategy is to show that $[A \rightarrow_\ell B]$ is a quotient algebra of $[A, B]$, which is a completely distributive Ω-lattice by Lemma 6.2.

Define $k : [A, B] \longrightarrow [A, B]$ by $k(f)(a) = \bigvee_{x\in A} \Downarrow(a)(x) \otimes f(x)$ for all $f \in [A, B]$ and $a \in A$, where, $\otimes$ denotes the tensor in B.

(1) $k : [A, B]_0 \longrightarrow [A, B]_0$ preserves order. Trivial by definition.

(2) $k(f) \leq f$ for all $f \in [A, B]$. Because $\Downarrow(a)(x) \leq A(x, a)$ for $a, x \in A$, then $\Downarrow(a)(x) \otimes f(x) \leq A(x, a) \otimes f(x) \leq f(a)$.

(3) k is idempotent. For all $f \in [A, B]$ and $a \in A$,

$$\begin{aligned}
k \circ k(f)(a) &= \bigvee_{x\in A} \Downarrow(a)(x) \otimes k(f)(x) \\
&= \bigvee_{x\in A} \Big[\Downarrow(a)(x) \otimes \Big(\bigvee_{y\in A} \Downarrow(x)(y) \otimes f(y) \Big) \Big] \\
&= \bigvee_{y\in A} \Big[\Big(\bigvee_{x\in A} \Downarrow(a)(x) * \Downarrow(x)(y) \Big) \otimes f(y) \Big] \\
&= \bigvee_{y\in A} \Downarrow(a)(y) \otimes f(y) \quad \text{(Prop. 4.3)} \\
&= k(f)(a).
\end{aligned}$$

(4) k preserves tensors.

$$k(\alpha \otimes f)(a) = \bigvee_{x\in A} [\Downarrow(a)(x) \otimes (\alpha \otimes f)(x)]$$

$$
\begin{aligned}
&= \bigvee_{x\in A}[\Downarrow(a)(x)\otimes(\alpha\otimes f(x))] \\
&= \bigvee_{x\in A}[\alpha\otimes(\Downarrow(a)(x)\otimes f(x))] \\
&= \alpha\otimes\bigvee_{x\in A}[\Downarrow(a)(x)\otimes f(x)] \\
&= \alpha\otimes k(f)(a) \\
&= (\alpha\otimes k(f))(a).
\end{aligned}
$$

(5) $k : [A,B]_0 \longrightarrow [A,B]_0$ preserves joins. Indeed, for all $a \in A$,

$$
\begin{aligned}
k\Big(\bigvee_{t\in T} f_t\Big)(a) &= \bigvee_{x\in A}\Big[\Downarrow(a)(x)\otimes\Big(\bigvee_{t\in T} f_t\Big)(x)\Big] \\
&= \bigvee_{x\in A}\Big[\Downarrow(a)(x)\otimes\bigvee_{t\in T} f_t(x)\Big] \\
&= \bigvee_{t\in T}\bigvee_{x\in A}\Big[\Downarrow(a)(x)\otimes f_t(x)\Big] \\
&= \bigvee_{t\in T} k(f_t)(a).
\end{aligned}
$$

(6) k is an Ω-functor. This follows from a combination of (1) and (4).

(7) $k(f) = f$ if and only if f is cocontinuous. Necessity is trivial by (4)-(6). Sufficiency: If f is cocontinuous, then for all $a \in A$,

$$
\begin{aligned}
f(a) &= f(\sup\Downarrow(a)) = \sup f(\Downarrow(a)) \\
&= \bigvee_{y\in B} f(\Downarrow(a))(y)\otimes y \quad \text{(Prop. \textbf{??}(1))} \\
&= \bigvee_{y\in B}\Big[\Big(\bigvee_{f(x)=y}\Downarrow(a)(x)\Big)\otimes y\Big] \\
&= \bigvee_{x\in A}\Downarrow(a)(x)\otimes f(x) \\
&= k(f)(a).
\end{aligned}
$$

Therefore, k is a cocontinuous kernel operator on $[A,B]$, the corresponding quotient algebra is $[A \to_\ell B]$ by (7). Hence $[A \to_\ell B]$ is completely distributive. $\Box$

A natural question is to ask whether the category of right adjoints between two completely distributive Ω-lattices is also completely distributive. At first, we say that for any Ω-categories A and B, the Ω-category $[B \to_r A]$ of right adjoints from B to A (as a subcategory of $[B,A]$) is isomorphic to the dual category of $[A \to_\ell B]$ consisting of left adjoints from A to B. To see this, we need only check that for any Ω-adjunctions (f_1, g_1) and (f_2, g_2),

$$
\bigwedge_{x\in A} B(f_1(x), f_2(x)) = \bigwedge_{y\in B} A(g_2(y), g_1(y)).
$$

Indeed, for any $y \in B$, let $x = g_2(y)$. Then

$$\begin{aligned} B(f_1(x), f_2(x)) &= B(f_1(x), f_2(g_2(y))) \leq B(f_1(x), y) \\ &= A(x, g_1(y)) = A(g_2(y), g_1(y)). \end{aligned}$$

Therefore,

$$\bigwedge_{x \in A} B(f_1(x), f_2(x)) \leq \bigwedge_{y \in B} A(g_2(y), g_1(y)).$$

Conversely, for any $x \in A$, let $y = f_2(x)$. Then

$$\begin{aligned} A(g_2(y), g_1(y)) &= A(g_2(f_2(x)), g_1(y)) \leq A(x, g_1(y)) \\ &= B(f_1(x), y) = B(f_1(x), f_2(x)). \end{aligned}$$

Therefore,

$$\bigwedge_{x \in A} B(f_1(x), f_2(x)) \geq \bigwedge_{y \in B} A(g_2(y), g_1(y)).$$

Particularly, if both A and B completely distributive Ω-lattices, then the dual category of the Ω-category of right adjoints from B to A, or that from A to B, is a completely distributive Ω-lattice. But, this does not mean that $[B \to_r A]$ is completely distributive since, as we shall see in the next section, the dual category of a completely distributive Ω-lattice is not necessarily a completely distributive Ω-lattice.

7 When Ω is a Girard quantale

In this section, we investigate the complete distributivity of the dual of a completely distributive Ω-lattice. The result shows that this depends heavily on the properties of Ω. That is, when Ω is an integral commutative quantale, then every completely distributive Ω-lattice is dually completely distributive if and only if Ω is a Girard quantale. This conclusion should be compared with the fact that in any topos $\mathcal{E}$, the dual of every constructive completely distributive lattice is constructive completely distributive if and only if $\mathcal{E}$ is a Boolean topos [28, 40].

By definition, it is easy to see that an Ω-category A is dually completely distributive, i.e., $A^{\rm op}$ is completely distributive, if and only if the Ω-functor $\inf : [A, \Omega]^{\rm op} \longrightarrow A$ has a right adjoint.

Example 7.1 Suppose that Ω is a commutative Girard quantale. Then the Ω-functor $\inf : [\Omega, \Omega]^{\rm op} \longrightarrow \Omega$ has a right adjoint. Thus $\Omega^{\rm op}$ is a completely distributive Ω-lattice. To this end, we show that the mapping $d : \Omega \longrightarrow [\Omega, \Omega]^{\rm op}$, given by $d(x)(t) = x \to 0$ for all $x \in \Omega,\ t \in \Omega$, is a right adjoint of $\inf : [\Omega, \Omega]^{\rm op} \longrightarrow \Omega$.

At first, for all $\psi \in [\Omega, \Omega]$, we have that

(1) ψ is increasing; and

(2) for any $t \in \Omega$, $\psi(0) \to 0 \leq \psi(t) \to t$.

To see (2), for any $t \in \Omega$,

$$\begin{aligned} t \to 0 \le \psi(t) \to \psi(0) &\iff \psi(t) \le (t \to 0) \to \psi(0) = (\psi(0) \to 0) \to t \\ &\iff \psi(0) \to 0 \le \psi(t) \to t. \end{aligned}$$

Therefore, for all $\psi \in [\Omega, \Omega]$,

$$\inf \psi = \bigwedge_{t \in \Omega} \psi(t) \to t = \psi(0) \to 0.$$

Hence, for all $\psi \in [\Omega, \Omega]$ and $x \in \Omega$,

$$\begin{aligned} \Omega(\inf \psi, x) &= (\psi(0) \to 0) \to x = (x \to 0) \to \psi(0) \\ &= (x \to 0) \to \bigwedge_{t \in \Omega} \psi(t) = \bigwedge_{t \in \Omega} (x \to 0) \to \psi(t) \\ &= [\Omega, \Omega]^{\mathrm{op}}(\psi, d(x)). \end{aligned}$$

That means, d is a right adjoint of inf.

Recall that an Ω-subset $\phi : A \longrightarrow \Omega$ of an Ω-category A is said to be finite if the set $\{x \in A \mid \phi(x) \neq 0\}$ is finite. Noticing that a complete lattice H is a complete Heyting algebra if and only if the supremum operator $\sup : \mathcal{D}(H) \longrightarrow H$ preserves finite meets, we introduce the following.

Definition 7.2 A complete Ω-lattice A is called a complete Ω-Heyting algebra if the supremum operator $\sup : [A^{\mathrm{op}}, \Omega] \longrightarrow A$ preserves finite infs.

Clearly, for a complete Ω-lattice A, the dual A^{op} is a complete Ω-Heyting algebra if and only if the Ω-functor $\inf : [A, \Omega]^{\mathrm{op}} \longrightarrow A$ preserves finite sups.

Every completely distributive Ω-lattice is a complete Ω-Heyting algebra because the supremum operator has a left adjoint, hence it preserves (all) infs. Consequently, the complete Ω-lattices $[A^{\mathrm{op}}, \Omega]$, $[\Omega^X]$ and $(\Omega, \to)$ are all complete Ω-Heyting algebras.

Proposition 7.3 *Suppose A is a complete Ω-lattice. Then, the followings are equivalent:*

(1) *A is a complete Ω-Heyting algebra.*

(2) $\sup : [A^{\mathrm{op}}, \Omega]_0 \longrightarrow A_0$ *preserves finite meets and* $\sup(\alpha \to \lambda) = \alpha \to \sup \lambda$ *for all* $\alpha \in \Omega, \lambda \in [A^{\mathrm{op}}, \Omega]$. □

Now we are at the position to prove the main result of this section, Theorem 1.1 stated in the introduction.

Proof of Theorem 1.1. (1) $\Rightarrow$ (2): Suppose Ω is a commutative Girard quantale. We claim at first that for any complete Ω-lattice L, there is an isomorphism $[L^{\mathrm{op}}, \Omega] \cong [L, \Omega]^{\mathrm{op}}$. In fact, define $\neg : [L^{\mathrm{op}}, \Omega] \longrightarrow [L, \Omega]^{\mathrm{op}}$ by $\neg\phi(x) = \phi(x) \to 0$ for

all $\phi \in [L^{\rm op}, \Omega]$ and $x \in A$. At first, $\neg$ is bijective by the law of double negation. Secondly, because

$$(\alpha \rightarrow 0) \rightarrow (\beta \rightarrow 0) = \beta \rightarrow ((\alpha \rightarrow 0) \rightarrow 0) = \beta \rightarrow \alpha$$

for all $\alpha, \beta \in \Omega$, we obtain that for all $\phi_1, \phi_2 \in [L^{\rm op}, \Omega]$,

$$\begin{aligned} [L^{\rm op}, \Omega](\phi_1, \phi_2) &= \bigwedge_{x \in L} \phi_1(x) \rightarrow \phi_2(x) \\ &= \bigwedge_{x \in L} (\phi_2(x) \rightarrow 0) \rightarrow (\phi_1(x) \rightarrow 0) \\ &= [L, \Omega]^{\rm op}(\phi_1 \rightarrow 0, \phi_2 \rightarrow 0). \end{aligned}$$

Therefore, $\neg : [L^{\rm op}, \Omega] \longrightarrow [L, \Omega]^{\rm op}$ is an Ω-isomorphism and $[L^{\rm op}, \Omega]^{\rm op}$ is a completely distributive Ω-lattice since $[L, \Omega]$ is completely distributive by Proposition 4.5.

Since L is completely distributive, $\sup : [L^{\rm op}, \Omega] \longrightarrow L$ is a complete Ω-lattice morphism. Consequently, $\sup^{\rm op} : [L^{\rm op}, \Omega]^{\rm op} \longrightarrow L^{\rm op}$ is also a complete Ω-lattice morphism. Thus, as a quotient algebra of a completely distributive Ω-lattice, $L^{\rm op}$ is completely distributive.

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Suppose $\Omega^{\rm op}$ is a complete Ω-Heyting algebra. Then the Ω-functor $\inf : [\Omega, \Omega]^{\rm op} \longrightarrow \Omega$ preserves finite sups. Particularly, inf preserves tensors. Therefore, for each $\alpha \in \Omega$,

$$\inf(\alpha \otimes \underline{0}) = \alpha * \inf \underline{0} = \alpha * 1 = \alpha,$$

where $\alpha \otimes \underline{0}$ denotes the tensor of α and the constant functor $\underline{0}$ in $[\Omega, \Omega]^{\rm op}$, or equivalently, the cotensor of α and $\underline{0}$ in $[\Omega, \Omega]$, that is, $\alpha \otimes \underline{0} = \alpha \rightarrow \underline{0}$. Meanwhile,

$$\inf(\alpha \rightarrow \underline{0}) = \bigwedge_{x \in A} ((\alpha \rightarrow 0) \rightarrow x) = (\alpha \rightarrow 0) \rightarrow 0.$$

Therefore, $\alpha = (\alpha \rightarrow 0) \rightarrow 0$. Hence, Ω is a commutative Girard quantale. $\square$

Corollary 7.4 *Suppose $(H, \wedge, \rightarrow)$ is a complete Heyting algebra. Then $(H, \rightarrow)^{\rm op}$ is an H-Heyting algebra if and only if H is a boolean algebra.* $\square$

Corollary 7.5 *Suppose Ω is a commutative Girard quantale, A and B are completely distributive Ω-lattices. Then the category of right adjoints $[A \rightarrow_r B]$ between A and B is a completely distributive Ω-lattices.* $\square$

We end this section with a conclusion on the existence of free completely distributive Ω-lattices when Ω is a commutative Girard quantale.

Proposition 7.6 *If Ω is a commutative Girard quantale, then the forgetful functor $G : \Omega\text{-}\mathbf{CD} \longrightarrow \mathbf{Set}$ has a left adjoint. Hence, for every set X, there is free completely distributive Ω-lattice generated by X.*

Proof. For each set X, let $F(X) = [[\Omega^X], \Omega]^{\mathrm{op}}$. Then $F(X)$ is completely distributive. Let $\eta_X : X \longrightarrow F(X)$ be given by $\eta_X(x)(\lambda) = \lambda(x)$ for all $x \in X$ and $\lambda \in \Omega^X$.

It suffices to show that for any completely distributive Ω-lattice A and any function $f : X \longrightarrow A$, there is a unique complete lattice morphism $g : F(X) \longrightarrow A$ such that $f = g \circ \eta_X$.

Existence: Since $f : X \longrightarrow A$, we have an Ω-functor $f^{\leftarrow} : [A^{\mathrm{op}}, \Omega] \longrightarrow [\Omega^X]$, and then an Ω-functor $(f^{\leftarrow})^{\leftarrow} : [[\Omega^X], \Omega]^{\mathrm{op}} \longrightarrow [[A^{\mathrm{op}}, \Omega], \Omega]^{\mathrm{op}}$.

Since $[A^{\mathrm{op}}, \Omega]$ is completely distributive, $[A^{\mathrm{op}}, \Omega]^{\mathrm{op}}$ is also completely distributive by Theorem 1.1. Therefore, the Ω-functor $\inf_{[A^{\mathrm{op}},\Omega]} : [[A^{\mathrm{op}}, \Omega], \Omega]^{\mathrm{op}} \longrightarrow [A^{\mathrm{op}}, \Omega]$ has a right adjoint, hence it is complete lattice morphism. Let g be the composition of the following functors:

$$[[\Omega^X], \Omega]^{\mathrm{op}} \xrightarrow{(f^{\leftarrow})^{\leftarrow}} [[A^{\mathrm{op}}, \Omega], \Omega]^{\mathrm{op}} \xrightarrow{\inf_{[A^{\mathrm{op}},\Omega]}} [A^{\mathrm{op}}, \Omega] \xrightarrow{\sup_A} A.$$

Then g is a complete lattice morphism. It remains to show that $f = g \circ \eta_X$. Indeed, for each $x \in X$ and $\lambda \in [A^{\mathrm{op}}, \Omega]$,

$$(f^{\leftarrow})^{\leftarrow})(\eta_X(x)(\lambda) = \eta_X(x)(f^{\leftarrow}(\lambda)) = \lambda(f(x)).$$

Thus,

$$\begin{aligned}
\inf (f^{\leftarrow})^{\leftarrow}(\eta_X(x)) &= \bigwedge_{\lambda \in [A^{\mathrm{op}},\Omega]} \Big((f^{\leftarrow})^{\leftarrow}(\eta_X(x))(\lambda) \to \lambda \Big) \\
&= \bigwedge_{\lambda \in [A^{\mathrm{op}},\Omega]} \Big(\lambda(f(x)) \to \lambda \Big) \\
&= \mathbf{y}(f(x)).
\end{aligned}$$

Consequently, $g \circ \eta_X(x) = \sup \mathbf{y}(f(x)) = f(x)$.

Uniqueness: At first, we show that for all $G \in [[\Omega^X], \Omega]$,

$$G = \bigvee_{\lambda \in \Omega^X} \Big[G(\lambda) * \Big(\bigwedge_{x \in X} (\lambda(x) \to \eta_X(x)) \Big) \Big].$$

Indeed, for all $\mu \in \Omega^X$,

$$\Big(\bigwedge_{x \in X} (\lambda(x) \to \eta_X(x)) \Big)(\mu) = \bigwedge_{x \in X} (\lambda(x) \to \mu(x)) = [\Omega^X](\lambda, \mu).$$

Thus,

$$\Big[\bigvee_{\lambda \in \Omega^X} \Big(G(\lambda) * \Big(\bigwedge_{x \in X} (\lambda(x) \to \eta_X(x)) \Big) \Big) \Big](\mu) = \bigvee_{\lambda \in \Omega^X} G(\lambda) * [\Omega^X](\lambda, \mu) = G(\mu).$$

Therefore, in $[[\Omega^X], \Omega]^{\mathrm{op}}$,

$$G = \bigwedge_{\lambda \in \Omega^X} \Big[G(\lambda) \rightarrowtail \Big(\bigvee_{x \in X} (\lambda(x) \otimes \eta_X(x)) \Big) \Big].$$

Suppose $g : F(X) \longrightarrow A$ is a complete lattice morphism with $f = g \circ \eta_X$. Then

$$g(G) = \bigwedge_{\lambda \in \Omega^X} \Big[G(\lambda) \rightarrowtail \Big(\bigvee_{x \in X} (\lambda(x) \otimes f(x)) \Big) \Big].$$

Consequently, g is unique. $\Box$

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